TRANSFORMASI LAPLACE
"The easy parts"

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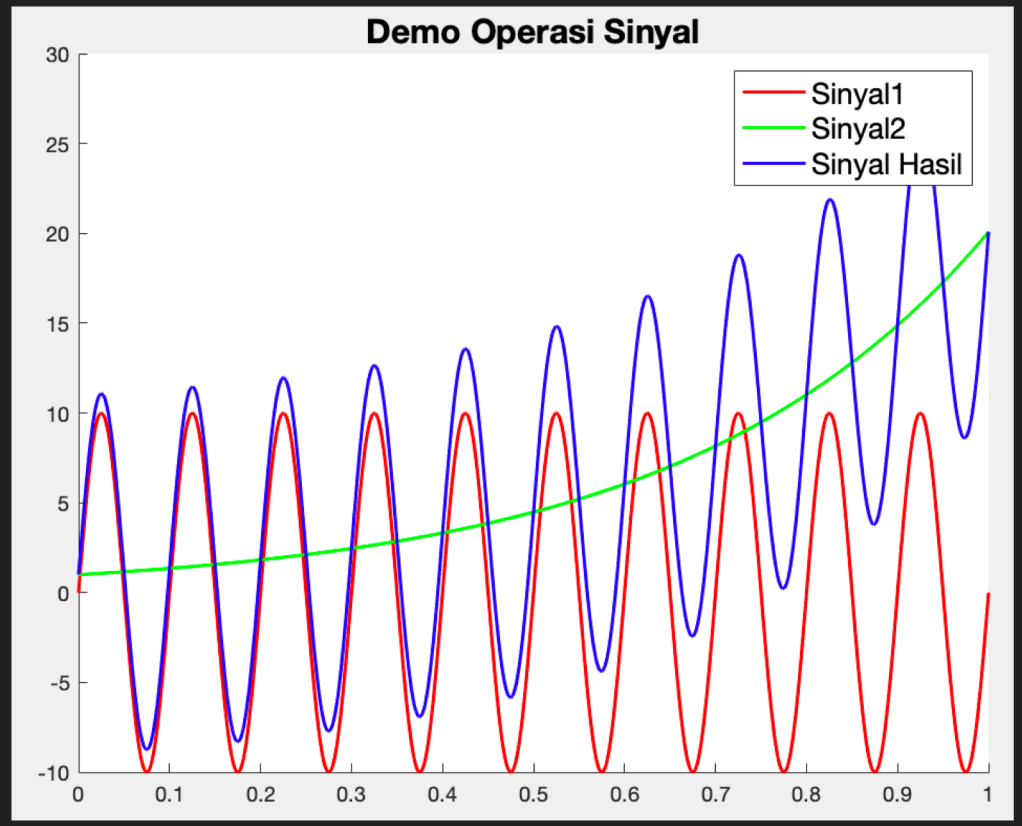
Definisi

Transformasi Laplace mengubah persamaan integral dan differensial menjadi sebuah persamaan aljabar dengan tujuan untuk memudahkan dalam menganalisis secara numerik.

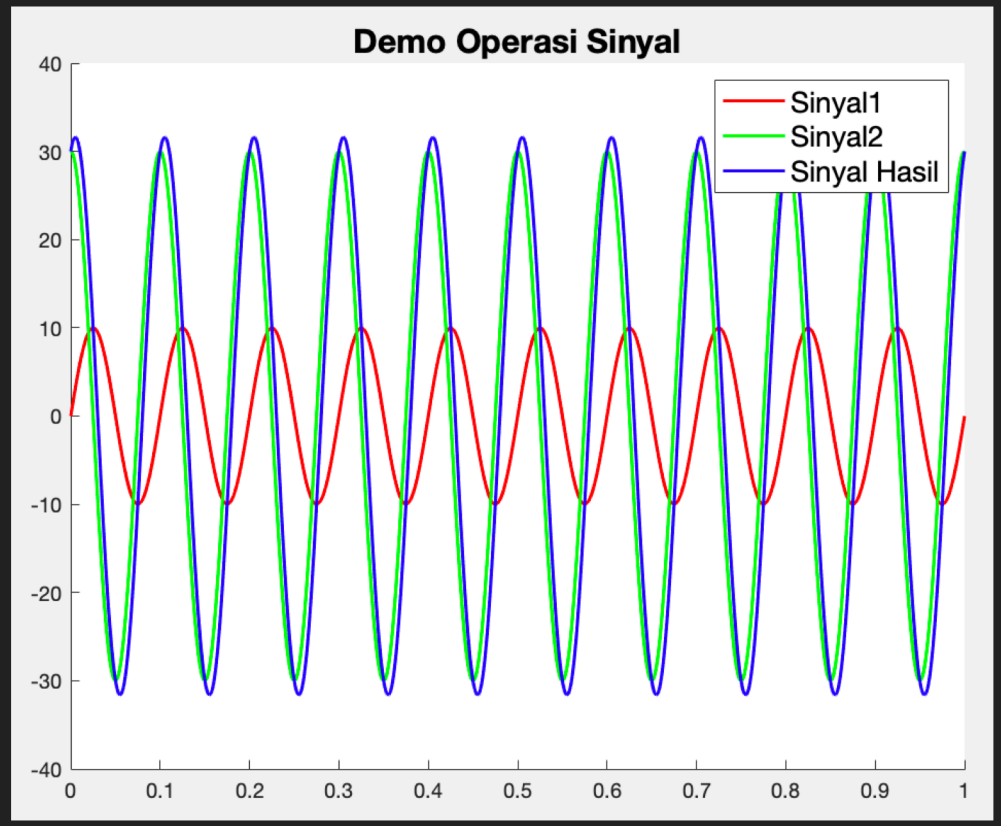
Transformasi Laplace mirip seperti fasor, akan tetapi :

1. Dapat diaplikasikan untuk semua sinyal tidak hanya sinyal sinus
2. Dapat digunakan untuk menangani kondisi yg “tidak stabil” – condong ke konsep sistem

Contoh!



a. Operasi penjumlahan sinyal sinus dan exponential



b. Operasi penjumlahan sinyal sinus dan sinus

TRANSFORMASI LAPLACE

Tranformasi Laplace dari signal f (fungsi) adalah fungsi $F = \mathcal{L}(f)$ yang didefinisikan

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Example 1 – Sinyal Eksponensial

Tentukan Laplace Transform dari : $f(t) = e^t$

$$F(s) = \int_0^\infty e^t \, e^{-st} \, dt = \int_0^\infty e^{(1-s)t} \, dt = \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty = \frac{1}{s-1}$$

Example 2 – Unit step

Tentukan Laplace Transform dari sinyal unit step : $f(t) = 1$ untuk $t \geq 0$

$$F(s) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

Example 3 : Sinyal Cosinus

Tentukan Laplace Transform dari : $f(t) = \cos \omega t$

$$f(t) = (1/2)e^{j\omega t} + (1/2)e^{-j\omega t}$$

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}\left((1/2)e^{j\omega t} + (1/2)e^{-j\omega t}\right)\, dt \\
&= (1/2)\int_0^\infty e^{(-s+j\omega)t}\, dt + (1/2)\int_0^\infty e^{(-s-j\omega)t}\, dt \\
&= (1/2)\frac{1}{s-j\omega} + (1/2)\frac{1}{s+j\omega} \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}$$

Example 3 : Sinyal Sinus

Tentukan Laplace Transform dari : $y(t) = \sin(\omega t)$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$Y(s) = \int_{0^-}^{\infty} \sin(\omega t) e^{-st} dt$$

$$Y(s) = \int_{0^-}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st} dt = \frac{1}{2j}\int_{0^-}^{\infty} e^{j\omega t} e^{-st} dt - \frac{1}{2j}\int_{0^-}^{\infty} e^{-j\omega t} e^{-st} dt$$

$$Y(s) = \frac{1}{2j}\frac{1}{s - j\omega} - \frac{1}{2j}\frac{1}{s + j\omega}$$

$$Y(s) = \frac{1}{2j}\frac{1}{(s - j\omega)}\frac{(s + j\omega)}{(s + j\omega)} - \frac{1}{2j}\frac{1}{(s + j\omega)}\frac{(s - j\omega)}{(s - j\omega)}$$

$$= \frac{1}{2j}\frac{(s + j\omega) - (s - j\omega)}{\left(s^2 - \cancel{sj\omega} + \cancel{sj\omega} - (j\omega)^2\right)} = \frac{1}{2j}\frac{2j\omega}{s^2 + \omega^2}$$

$$= \frac{\omega}{s^2 + \omega^2}$$

$$\sin(\omega t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega}{s^2 + \omega^2}$$

Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$

STUDI KASUS 1

Diketahui :

$$f(t) = 3e^t$$

$$g(t) = 2e^t$$

Tentukan nilai penjumlahan dari kedua sinyal tersebut (nilai h(t) dengan menggunakan Laplace Transform!

$$h(t) = f(t) + g(t)$$

Solusi

Tahap 1 : $f(t) = 3e^t \rightarrow F(s) = \frac{3}{1-s}$

Tahap 2 : $g(t) = 2e^t \rightarrow G(s) = \frac{2}{1-s}$

Tahap 3 : $h(t) = f(t) + g(t) \rightarrow \mathrm{H}(s) = F(s) + G(s)$

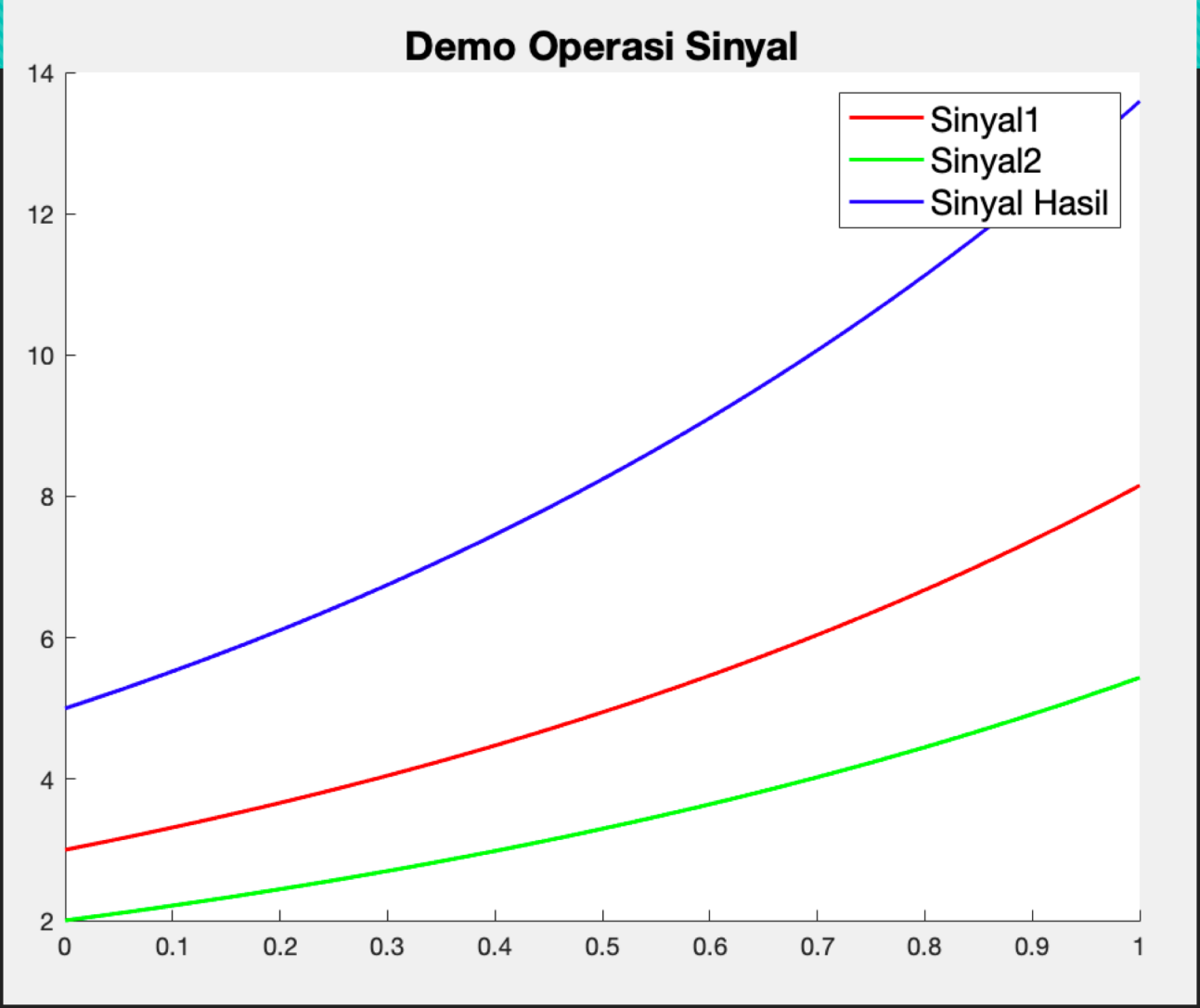
$$H(s) = \frac{3}{1-s} + \frac{2}{1-s} = \frac{5}{1-s}$$

Lanjutan (Inverse Laplace)

$$H(s) = \frac{3}{1-s} + \frac{2}{1-s} = \frac{5}{1-s}$$

$$H(s) = \frac{5}{1-s} \quad \xrightarrow{\text{Inverse Laplace}} \quad h(t) = 5e^{t}$$

Bukti (Grafik)



$Sinyal\ 1\ yaitu\ f(t) = 3e^t$

$Sinyal\ 2\ yaitu\ g(t) = 2e^t$

$Sinyal\ hasil\ yaitu$

$h(t) = 5e^t$

TERBUKTI VALID!

STUDI KASUS 2

Diketahui :

$$f(t) = 10\sin(2\pi 10t)$$

$$g(t) = 5\sin(2\pi 10t)$$

Tentukan nilai penjumlahan dari kedua sinyal tersebut (nilai h(t) dengan menggunakan Laplace Transform!

$$h(t) = f(t) + g(t)$$

Solusi

$$\omega = 2pi10$$

Tahap 1 : $f(t) = 10\sin(\omega t)$ → $F(s) = \frac{10a}{s^2 + a^2}$

Tahap 2 : $g(t) = 5\sin(\omega t)$ → $G(s) = \frac{5a}{s^2 + a^2}$

Tahap 3 : $h(t) = f(t) + g(t)$ → $\mathrm{H}(s) = F(s) + G(s)$

$$H(s) = \frac{10a}{s^2+a^2} + \frac{5a}{s^2+a^2} = \frac{15a}{s^2+a^2}$$

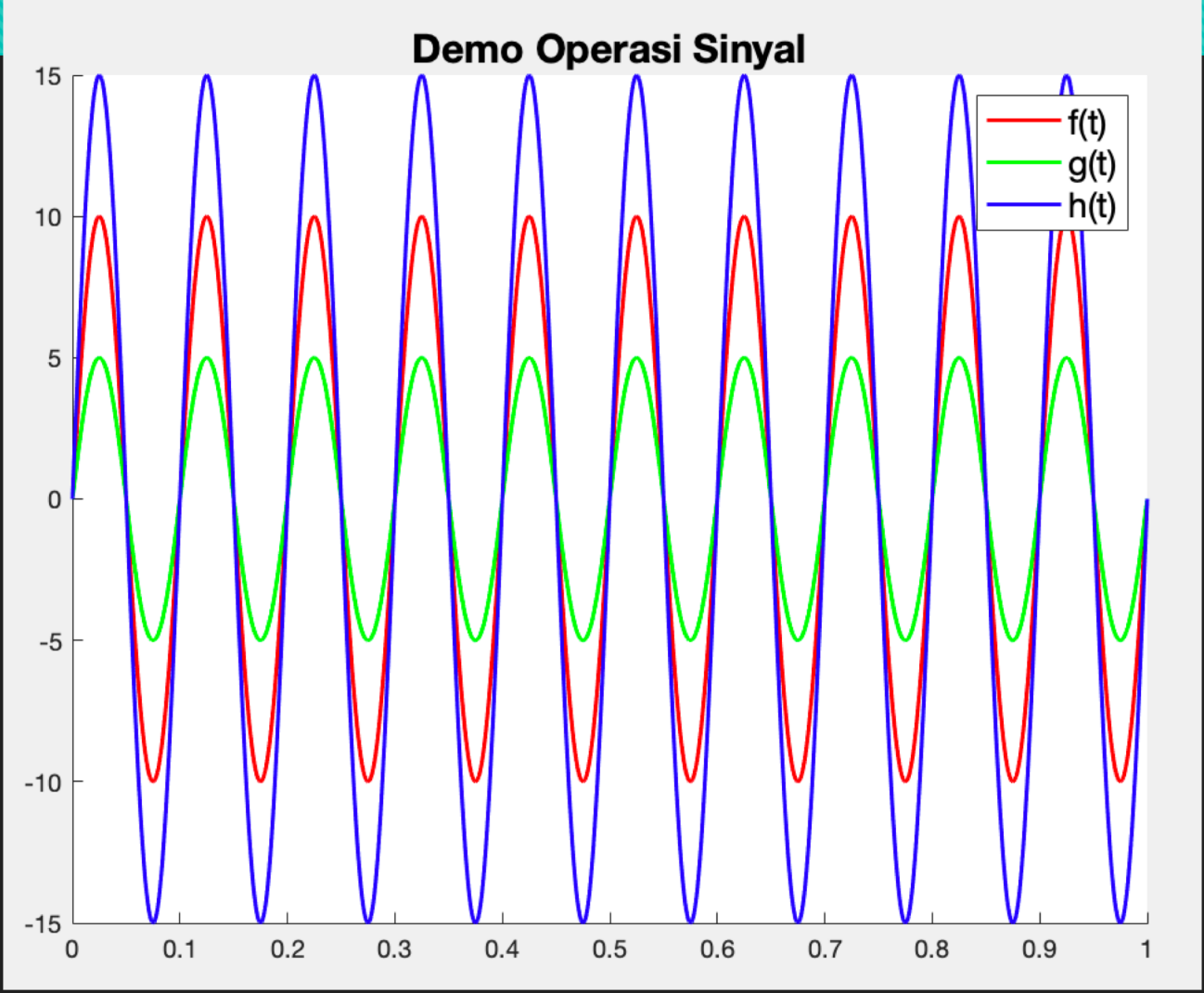
Lanjutan (Inverse Laplace)

$$H(s) = \frac{10a}{s^2-a^2} + \frac{5a}{s^2-a^2} = \frac{15a}{s^2+a^2}$$

$$H(s) = \frac{15a}{s^2 + a^2}$$ Inverse Laplace $$h(t) = 15 \sin \omega t$$

Dimana $\omega = 2pi10$ maka $h(t) = 15 \sin(2pi10t)$

Bukti (Grafik)



$$f(t) = 10\sin(2\pi 10t)$$

$$g(t) = 5\sin(2\pi 10t)$$

$$h(t) = f(t) + g(t)$$

$$h(t) = 15\sin 2\pi 10t$$

TERBUKTI VALID!

Semoga jelas dan paham!