10. A company can use two types of machine A and B, in a manufacturing plant. The number of operators required and the running cost per day are given as

	Cost per day	Available operators	Floor area (m^2)	Profit per machine
Machine A	6	2	2	20
Machine B	3	4	2	30
Maximum available	360	280	160	

(a) Write down the inequality constraints and the profit function.
(b) Graph the inequalities. From the graph determine the number of machines A and B which should be used to maximise profits.
(c) Confirm your answer in (b) algebraically.

11. A gardener requires fertiliser to have a minimum of 80 units of nitrogen, 15 units of potassium and 10 units of iron. Two fertiliser mixes are available.
Brand X contains 20, 2 and 1 units of the minerals while
Brand Y contains 4, 1, 2 units of the minerals respectively.
Brand X costs £18 per kg while brand Y costs £6 per kg.
(a) Write down the inequality constraints and the equation of the cost function.
(b) Graph the inequality constraints. Hence determine the number of kg of each brand which provide the minimum mineral requirements at minimum cost.
(c) Confirm your answer in (b) algebraically.

9.2 Matrices

At the end of this section you should be able to:

- Define matrices
- Add, subtract and transpose matrices
- Multiply matrices
- Apply matrix arithmetic
- Use matrix arithmetic to simplify calculations for large arrays of data.

Matrices are rectangular arrays of numbers or symbols. The usual arithmetic operations—addition, subtraction and multiplication—are performed with these arrays. However, dealing with arrays of numbers, and not just single numbers as in ordinary arithmetic, means that there will be restrictions on all these arithmetic operations. Since matrices and determinants (see next section) are both arrays of numbers, much of the terminology used to describe matrices is the same as that used in determinants such as minors, cofactor, dimension.

9.2.1 Matrices: definition

Matrices are rectangular arrays of numbers or symbols.

The dimension of a matrix is stated as the number of rows by number of columns. The following are examples of matrices, with their dimensions:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 2 & 5 \end{pmatrix} \qquad D = (-2 \quad 1 \quad 4)$$

Dimension 2×2 3×2 1×3

❒ Special matrices

The null matrix is a matrix of any dimension in which every element is zero, such as

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (0 \quad 0 \quad 0 \quad 0)$$

Dimension 2×2 3×1 1×4

The unit matrix or identity matrix is any square matrix in which every element is zero, except the elements on the main diagonal, each of which has the value 1, such as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The unit or identity matrix is represented by the symbol I.

Matrices are equal if they are of the same dimension and the corresponding elements are identical:

$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$ are equal, but the matrices

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ are not equal. The elements are identical, but the dimensions are not.

The transpose of a matrix is the matrix obtained by writing the rows of any matrix as columns or vice versa as follows:

$$\begin{array}{l} \\ \text{row 1} \rightarrow \\ \text{row 2} \rightarrow \end{array} \begin{array}{c} \text{row 1} \quad \text{row 2} \\ \downarrow \quad \downarrow \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \end{array}$$

where the superscript T indicates that the matrix is to be transposed.

9.2.2 Matrix addition and subtraction

To add matrices, add the corresponding elements, for example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \tag{9.10}$$

To subtract matrices, subtract the corresponding elements, for example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & b-f \\ c-g & d-h \end{pmatrix} \tag{9.11}$$

WORKED EXAMPLE 9.3

ADDING AND SUBTRACTING MATRICES

Given the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} \tag{9.12}$$

(a) Calculate
  (i) $A + C$
  (ii) $A - C$

(b) Why is it not possible to calculate $A + B$ or $A - B$?
Hence, state the restrictions on matrix addition and subtraction.

Solution

(a) (i) $$\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1+3 & 2+(-2) \\ -2+5 & 4+0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 4 \end{pmatrix}$$

(ii) $$\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1-3 & 2-(-2) \\ -2-5 & 4-0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -7 & 4 \end{pmatrix}$$

(b) When attempting to add the corresponding elements of the matrices A and B it is found that there is no third column in matrix A, therefore it is not possible to add pairs of corresponding elements in the two matrices. Matrix addition is not possible.

$$\begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+2 & ?+2 \\ -2+1 & 4+0 & ?+5 \end{pmatrix}$$

No corresponding elements to add!

The same problem arises when attempting to subtract matrices A and B: there is no third column in matrix A from which to subtract the elements in column three of matrix B.

To add or subtract two or more matrices, all matrices must have exactly the same dimensions

Scalar multiplication

A scalar is an ordinary number: 2, 5, −8, etc. When a matrix is multiplied by a scalar, each element in the matrix is multiplied by the scalar:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \tag{9.13}$$

WORKED EXAMPLE 9.4

MULTIPLICATION OF A MATRIX BY A SCALAR

(a) Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$$

calculate $5A$.
(b) Calculate $3I$, where I is the 3×3 unit matrix.

Solution

(a)
$$5A = \begin{pmatrix} 5 \times 1 & 5 \times 2 \\ 5 \times (-2) & 5 \times 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ -10 & 20 \end{pmatrix}$$

(b)
$$3I = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

9.2.3 Matrix multiplication

Two matrices A and B are multiplied as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix} \tag{9.14}$$

where row 1, row 2 are from matrix A, col. 1, col. 2 from matrix B.

To multiply a row by a column, for example (row 1 × col. 1), proceed as follows:

$$\begin{pmatrix} a & b \\ - & - \end{pmatrix} \times \begin{pmatrix} e & - \\ g & - \end{pmatrix} = \begin{pmatrix} (a \times e) + (b \times g) & - \\ - & - \end{pmatrix} \tag{9.15}$$

that is, from row 1 and column 1, multiply the first pair of elements in each, the second pair of elements in each, then add the products of these pairs.

The product of the two matrices written out in full is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \qquad (9.16)$$

dimension: $(2 \times 2) \times (2 \times 2) \quad = (2 \times 2)$

WORKED EXAMPLE 9.5

MATRIX MULTIPLICATION

Given the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix}$$

evaluate: (a) AC (b) CA (c) AB (d) BA

Compare your answers for (a) and (b).

Solution

(a) The product of the matrices A and C is given in general as:

$$AC = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix}$$

$$= \begin{pmatrix} (1)(3)+(2)(5) & (1)(-2)+(2)(0) \\ (-2)(3)+(4)(5) & (-2)(-2)+(4)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -2 \\ 14 & 4 \end{pmatrix}$$

(b) Similarly, CA is calculated as follows:

$$CA = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix}$$

$$= \begin{pmatrix} (3)(1)+(-2)(-2) & (3)(2)+(-2)(4) \\ (5)(1)+(0)(-2) & (5)(2)+(0)(4) \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -2 \\ 5 & 10 \end{pmatrix}$$

In the above example

$$AC = \begin{pmatrix} 13 & -2 \\ 14 & 4 \end{pmatrix}$$

and

$$CA = \begin{pmatrix} 7 & -2 \\ 5 & 10 \end{pmatrix}$$

therefore $AC \neq CA$. (Since two matrices are equal only if all corresponding elements are identical.)

So in matrix multiplication, the order of multiplication is important. In general, in matrix multiplication, $AC \neq CA$.

(c) The product of the matrices A and B is given in general as

$$AB = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} & \text{row 1} \times \text{col. 3} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} & \text{row 2} \times \text{col. 3} \end{pmatrix}$$

dimension: $(2 \times 2) \times (2 \times 3) = (2 \times 3)$

Multiplying the rows by the columns,

$$AB = \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(0)+(2)(1) & (1)(2)+(2)(0) & (1)(2)+(2)(5) \\ (-2)(0)+(4)(1) & (-2)(2)+(4)(0) & (-2)(2)+(4)(5) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 12 \\ 4 & -4 & 16 \end{pmatrix}$$

(d) To evaluate BA, proceed as usual:

$$BA = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \text{row 1} \times \text{col. 1} & \text{row 1} \times \text{col. 2} \\ \text{row 2} \times \text{col. 1} & \text{row 2} \times \text{col. 2} \end{pmatrix}$$

dimension $(2 \times 3) \times (2 \times 2)$

Then multiplying the rows in the first matrix by the columns in the second matrix, it is found that:

$$BA = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ -2 & 4 \end{pmatrix}$$

no third element in the columns of A to multiply the third elements in the rows of B

$$= \begin{pmatrix} (0)(1)+(2)(-2)+(2)(?) & (0)(2)+(2)(4)+(2)(?) \\ (1)(1)+(0)(-2)+(5)(?) & (1)(2)+(0)(4)+(5)(?) \end{pmatrix}$$

So the rows by the columns cannot be multiplied, since the rows in B contain three elements and the columns in A only two elements. Therefore, matrix multiplication is not possible. In general,

> Matrix multiplication BA is possible if the number of elements in the rows of the first matrix (B) is the same as the number of elements in the columns of the second matrix (A).

This condition for matrix multiplication can be established quickly by writing down the dimensions of the matrices to be multiplied, in order:

$A \times B$ = product

dimension of product: 2×3

dimensions: $(2 \times 2) \times (2 \times 3) = (2 \times 3)$

the same, so multiplication is possible

The 'inside' numbers are the same, therefore multiplication is possible. The 'outside' numbers give us the dimension of the product.

9.2.4 Applications of matrix arithmetic

WORKED EXAMPLE 9.6

APPLICATIONS OF MATRIX ARITHMETIC

A distributor records the weekly sales of personal computers (PCs) in three retail outlets in different parts of the country (see Table 9.5). The cost price of each model is:

Table 9.5 Number of computers sold in each shop

	Pentium (basic)	Pentium (extra)	Pentium (latest)
Shop A	150	320	180
Shop B	170	420	190
Shop C	201	63	58

Pentium (basic) £480, Pentium (extra) £600, Pentium (latest) £1020

The retail price of each model in each of the three shops is given in Table 9.6.

Table 9.6 Selling price of computers in each shop

	Pentium (basic)	Pentium (extra)	Pentium (latest)
Shop A	560	750	1580
Shop B	520	690	1390
Shop C	590	720	1780

Use matrix multiplication to calculate

(a) The total weekly cost of computers to each shop.
(b) The total weekly revenue for each model for each shop.
(c) The total weekly profit for each shop.

Which shop makes the greatest overall profit?

Solution

(a) The numbers sold from Table 9.5 may be written as a matrix, Q:

$$Q = \begin{matrix} \text{PC} \rightarrow \\ \text{Shop A} \\ \text{Shop B} \\ \text{Shop C} \end{matrix} \begin{matrix} \text{basic} \\ \\ \\ \end{matrix} \begin{matrix} \text{extra} \\ \\ \\ \end{matrix} \begin{matrix} \text{latest} \\ \\ \\ \end{matrix} \begin{pmatrix} 150 & 320 & 180 \\ 170 & 420 & 190 \\ 201 & 63 & 58 \end{pmatrix}$$

Write the cost of each type of computer as a column matrix:

$$C = \begin{pmatrix} 480 \\ 600 \\ 1020 \end{pmatrix}$$

If this cost matrix C is premultiplied by the numbers sold matrix, Q, the product will be a 3×1 matrix, in which the elements in each row gives the total cost of computers to each shop:

$$Q \cdot C = \text{Total cost}$$

$$\text{dimension:} \quad (3 \times 3)(3 \times 1) = (3 \times 1)$$

$$\begin{pmatrix} 150 & 320 & 180 \\ 170 & 420 & 190 \\ 201 & 63 & 58 \end{pmatrix} \begin{pmatrix} 480 \\ 600 \\ 1020 \end{pmatrix} = \begin{pmatrix} 150(480) + 320(600) + 180(1020) \\ 170(480) + 420(600) + 190(1020) \\ 201(480) + 63(600) + 58(1020) \end{pmatrix}$$

$$= \begin{pmatrix} 447\,600 \\ 527\,400 \\ 193\,440 \end{pmatrix}$$

Cost to: shop A = £447 600, shop B = £527 400, shop C = £193 440.

(b) The total revenue = price × quantity. The quantities are given by the matrix, Q, for the data in Table 9.5. The prices are obtained from the data in Table 9.6. Matrix multiplication, however, is carried out by multiplying rows by columns; therefore in order to multiply quantity × price for each PC, the rows in Table 9.6 must be written as columns in the prices matrix. That is, the prices matrix must be transposed before premultiplying by the quantities matrix, Q.

$$P = \begin{pmatrix} 560 & 750 & 1580 \\ 520 & 690 & 1390 \\ 590 & 720 & 1780 \end{pmatrix}^{\mathrm{T}} = \begin{matrix} \text{Price of basic PC} \rightarrow \\ \text{Price of extra PC} \rightarrow \\ \text{Price of latest PC} \rightarrow \end{matrix} \begin{pmatrix} 560 & 520 & 590 \\ 750 & 690 & 720 \\ 1580 & 1390 & 1780 \end{pmatrix}$$

Now multiply Q (quantities matrix) by P (prices matrix from data in Table 9.6):

$$
\begin{array}{lcc}
\text{PC type} & & \begin{array}{ccc} \text{basic} & \text{extra} & \text{latest} \\ \downarrow & \downarrow & \downarrow \end{array} \\
Q \times P = & \begin{array}{c} \text{shop A} \\ \text{shop B} \\ \text{shop C} \end{array} & \begin{pmatrix} 150 & 320 & 180 \\ 170 & 420 & 190 \\ 201 & 63 & 58 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{rc}
\begin{array}{r} \text{Price of basic} \rightarrow \\ \times\ \text{Price of extra} \rightarrow \\ \text{Price of latest} \rightarrow \end{array} & \begin{pmatrix} 560 & 520 & 590 \\ 750 & 690 & 720 \\ 1580 & 1390 & 1780 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{lcc}
Q \times P & \begin{array}{c} \text{shop A} \\ \downarrow \end{array} & \begin{array}{c} \text{shop B} \\ \downarrow \end{array} \\
= & \begin{pmatrix} 150(560) + 320(750) + 180(1580) & \text{N/A} \\ \text{N/A} & 170(520) + 420(690) + 190(1390) \\ \text{N/A} & \text{N/A} \end{pmatrix} &
\end{array}
$$

$$
\begin{array}{cc}
\begin{array}{c} \text{shop C} \\ \downarrow \end{array} & \\
\begin{array}{c} \text{N/A} \\ \times \quad \text{N/A} \\ 201(590) + 63(720) + 58(1780) \end{array} & \Bigg)
\end{array}
$$

The total revenue for each shop is given by the elements on the main diagonal of the product matrix. Total revenue for shops A, B and C are summarised in a column matrix, TR;

$$
\begin{array}{rcl}
 & \begin{array}{ccc} \text{Basic PC} & \text{Extra PC} & \text{Latest PC} \end{array} & \\
\text{Revenue matrix,} \quad TR = & \begin{pmatrix} 84\,000 + 240\,000 + 284\,400 \\ 88\,400 + 289\,800 + 264\,100 \\ 118\,590 + 45\,360 + 103\,240 \end{pmatrix} & \begin{array}{l} \leftarrow \text{shop A} \\ \leftarrow \text{shop B} \\ \leftarrow \text{shop C} \end{array} = \begin{pmatrix} 604\,400 \\ 642\,300 \\ 267\,190 \end{pmatrix}
\end{array}
$$

(c) Profit $= TR - TC$

$$
= \begin{pmatrix} 604\,400 \\ 642\,300 \\ 267\,190 \end{pmatrix} - \begin{pmatrix} 447\,600 \\ 527\,400 \\ 193\,440 \end{pmatrix} = \begin{pmatrix} 156\,800 \\ 114\,900 \\ 73\,750 \end{pmatrix} \begin{array}{l} \leftarrow \text{shop A} \\ \leftarrow \text{shop B} \\ \leftarrow \text{shop C} \end{array}
$$

So shop A makes the highest profit.

Progress Exercises 9.2 Introductory Matrix Algebra and Applications

1. Given the matrices: $A = \begin{pmatrix} 1 & -4 \\ 0 & -9 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 3 \\ -7 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 5 & -1 & -1 \\ 12 & 0 & 2 \end{pmatrix}$

calculate, if possible:

(a) $A + B$ (b) $A - B$ (c) $A + 4B$ (d) $A + I$ (e) AI (f) $A + C$
(g) $A + B^{\mathrm{T}}$ (h) BC (i) CB (j) CB^{T} (k) $(AB)^{\mathrm{T}}$ (l) $C + 5I$
(m) $C^{\mathrm{T}}A$ (n) $(BC)^{\mathrm{T}}$ (o) $AC + B$

2. Given the matrices: $A = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 2 & -5 \\ -1 & 1 \end{pmatrix}$

Determine each of the following, if possible:

(a) $A + C$ (b) $B^{\mathrm{T}} + C$ (c) BC (d) $A^{\mathrm{T}}B^{\mathrm{T}}$ (e) $B^{\mathrm{T}}A^{\mathrm{T}}$

3. A fast-food chain has three shops, A, B and C. The average daily sales and profit in each shop is given in the following table:

	Units sold			**Units profit**		
	Shop A	Shop B	Shop C	Shop A	Shop B	Shop C
Burgers	800	400	500	20p	40p	33p
Chips	950	600	700	50p	45p	60p
Drinks	500	1200	900	30p	35p	20p

Use matrix multiplication to determine

(a) The profit for each product
(b) The profit for each shop

4. The percentage of voters who will vote for party candidates A, B, and C is given in the following table:

	A	B	C	No. of voters
Area 1	60%	20%	20%	25 000
Area 2	45%	30%	25%	60 000
Area 3	38%	30%	32%	98 000

Use matrix multiplication to calculate the total number of votes for each candidate.

5. Given the following matrices:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

(a) Show that $AB \neq BA$.
(b) Determine the following if possible:

(i) AC (ii) AD (iii) DC (iv) DCC^T

9.3 Solution of Equations: Elimination Methods

In Chapter 3, simultaneous equations were solved by adding multiples of equations to other equations until one equation in one variable was obtained. With one variable known, the remaining variables were easily found. See Worked Examples 3.1, 3.2, 3.3, 3.6. Worked Example 3.6 will be solved here by the method of Gaussian elimination.

9.3.1 Gaussian elimination

In its simplest form, Gaussian elimination is a technique for solving a system of n linear equations in n unknowns by systematically adding multiples of equations to other equations in such a way that we end up with a series of n equations, each containing one less unknown than the previous equation, with the last equation containing just one unknown. In other words, a set of equations such as

$$\begin{aligned} x + y - z &= 3 \\ 2x + y - z &= 4 \\ 2x + 2y + z &= 12 \end{aligned} \tag{9.17}$$

may be reduced systematically to

$$\begin{aligned} x + y - z &= 3 \\ -y + z &= 2 \\ 2z &= 4 \end{aligned} \tag{9.18}$$

The solutions to both sets of equations (9.17) and (9.18) are identical, but the reduced set of equations is easily solvable. The last equation contains just one unknown, z, so solve this first. Substitute this value of z into the middle equation to solve for y. Finally, with y and z known, solve the first equation for x. This is called solving by back substitution. The following worked example will demonstrate the method of Gaussian elimination and solving by back substitution.

WORKED EXAMPLE 9.7

SOLUTION OF A SYSTEM OF EQUATIONS: GAUSSIAN ELIMINATION

Solve the following equations by Gaussian elimination:

$$x + y - z = 3 \quad (1)$$
$$2x + y - z = 4 \quad (2)$$
$$2x + 2y + z = 12 \quad (3)$$

Solution

All three equations must be written in the same format: variables x, y, z (in the same order) on the LHS and constants on the RHS. With the equations written in order, the variables may be dropped in the calculations which follow.

$$\begin{array}{lll} x+y-z=3 & (1) & \\ 2x+y-z=4 & (2) & \rightarrow \\ 2x+2y+z=12 & (3) & \end{array} \quad \begin{array}{c} x\ \ y\ \ z\ \ \text{RHS} \\ \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & 1 & -1 & 4 \\ 2 & 2 & 1 & 12 \end{pmatrix} \end{array} \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

The 3×4 matrix is called the augmented matrix, A. The aim of Gaussian elimination is to systematically reduce this array of numbers to upper triangular form, similar in form to the reduced set of equations (9.18). A matrix of upper triangular form has zeros beneath the main diagonal (tinted background):

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \end{pmatrix}$$

Therefore, in column 1, we require zeros beneath the first element, and in column 2 we require zeros beneath the second element.

Action	Augmented matrix	Calculations
To get the required 0s, you must add (−2 × row 1) to row 2 and (−2 × row 1) to row 3	$\begin{array}{c} x\ \ y\ \ z\ \ \text{RHS} \\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \\ \begin{pmatrix} 1 & 1 & -1 & 3 \\ ② & 1 & -1 & 4 \\ ② & 2 & 1 & 12 \end{pmatrix} \end{array} \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$	[1] Updated once [2] Updated twice
row 2 + (−2 × row 1) →	$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 2 & 2 & 1 & 12 \end{pmatrix} \begin{array}{l} (1) \\ (2)^1 \\ (3) \end{array}$	$\begin{array}{rrrrl} -2 & -2 & 2 & -6 & (1) \times -2 \\ 2 & 1 & -1 & 4 & (2) \\ \hline 0 & -1 & 1 & 2 & \text{adding} \end{array}$
row 3 + (−2 × row 1) →	$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 3 & 6 \end{pmatrix} \begin{array}{l} (1) \\ (2)^1 \\ (3)^1 \end{array}$	$\begin{array}{rrrrl} -2 & -2 & 2 & -6 & (1) \times -2 \\ 2 & 2 & 1 & 12 & (3) \\ \hline 0 & 0 & 3 & 6 & \text{adding} \end{array}$

No further elimination is necessary, since there is a 0 beneath the second element in column 2. However, divide row 3 by 3 to simplify (optional)

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{array}{l} (1) \\ (2)^1 \\ (3)^2 \end{array} \leftarrow \text{divide row } 3^1 \text{ by } 3$$

Rewrite the equations from the augmented matrix:

$$\begin{aligned} x + y - z &= 3 \qquad (1) \\ -y + z &= -2 \qquad (2)^1 \\ z &= 2 \qquad (3)^2 \end{aligned} \tag{9.19}$$

Solve by back substitution

Start with equation $(3)^2$

$(3)^2 \rightarrow z = 2$

Substitute $z = 2$ into equation $(2)^1$

$(2)^1 \rightarrow -y + z = -2$ hence $-y + 2 = -2 \rightarrow y = 4$

Substitute $z = 2$ and $y = 4$ into equation $(3)^2$

$(3)^2 \rightarrow x + y - z = 3$ hence $x + 4 - 2 = 3 \rightarrow x = 1$

Solution: $x = 1$, $y = 4$, $z = 2$

In Worked Example 9.7 the elimination worked out quickly and easily; the numbers in the augmented matrix made the arithmetic easy. However, when the numbers are not so convenient, divide the row that is being used in the elimination by the value of the eliminating element, as illustrated in Worked Example 9.8.

WORKED EXAMPLE 9.8

MORE GAUSSIAN ELIMINATION

Solve the following equations by Gaussian elimination:

$$\begin{aligned} 2x + y + z &= 12 \\ 6x + 5y - 3z &= 6 \\ 4x - y + 3z &= 5 \end{aligned}$$

Solution

The equations are already written in the required form: variables in order on the LHS and constants on the RHS. Therefore, write down the augmented matrix and proceed with the Gaussian elimination:

$$\begin{aligned} 2 + y + z &= 12 \\ 6x + 5y - 3z &= 6 \\ 4x - y + 3z &= 5 \end{aligned} \quad \rightarrow \quad \begin{array}{c} \begin{array}{cccc} x & y & z & \text{RHS} \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 12 \\ 6 & 5 & -3 & 6 \\ 4 & -1 & 3 & 5 \end{pmatrix} \end{array}$$

Now proceed with the elimination on the augmented matrix.

Action	Augmented matrix					Calculations					
	x	y	z	RHS							
Make the tinted 2 into a 1 by dividing row 1 by 2	2	1	1	12	(1)	[1] Updated once					
	6	5	−3	6	(2)	[2] Updated twice					
	4	−1	3	5	(3)						
	1	0.5	0.5	6	$(1)^1$						
	6	5	−3	6	(2)						
	4	−1	3	5	(3)						
row 2 + (−6 × row 1^1)	1	0.5	0.5	6	$(1)^1$	−6	−3	−3	−36	$(1)^1 \times -6$	
	0	2	−6	−30	$(2)^1$	6	5	−3	6	(2)	
	4	−1	3	5	(3)	0	2	−6	30	adding	
row 3 + (−4 × row 1^1)	1	0.5	0.5	6	$(1)^1$	−4	−2	−2	−24	$(1)^1 \times -4$	
	0	2	−6	−30	$(2)^1$	4	−1	3	5	(3)	
	0	−3	1	−19	$(3)^1$	0	−3	1	−19	adding	
Make the tinted 2 into a 1 by dividing row 2^1 by 2	1	0.5	0.5	6	$(1)^1$						
	0	2	−6	−30	$(2)^1$						
	0	−3	1	−19	$(3)^1$						
	1	0.5	0.5	6	$(1)^1$						
	0	1	−3	−15	$(2)^2$						
	0	−3	1	−19	$(3)^1$						
row 3^1 + (3 × row 2^2)	1	0.5	0.5	6	$(1)^1$	0	3	−9	−45	$(2)^2 \times 3$	
	0	1	−3	−15	$(2)^2$	0	−3	1	−19	$(3)^1$	
	0	0	−8	−64	$(3)^2$	0	0	−8	−64	adding	

The elimination is now complete; solve by back substitution.
From equation $(3)^2$

$$-8z = -64 \quad \text{hence} \quad z = \frac{-64}{-8} = 8$$

Substitute $z = 8$ into equation $(2)^2$

$$y - (3 \times 8) = -15 \quad \text{hence} \quad y = -15 + (3 \times 8) = 9$$

Substitute $z = 8$ and $y = 9$ into equation $(1)^1$

$$x + (0.5 \times 9) + (0.5 \times 8) = 6 \quad \text{hence} \quad x = 6 - 4.5 - 4 = -2.5$$

Solution: $x = -2.5$, $y = 9$, $z = 8$

9.3.2 Gauss–Jordan elimination

Gauss–Jordan elimination goes further than Gaussian elimination, producing an augmented matrix with a main diagonal of ones:

$$\begin{pmatrix} 1 & 0 & 0 & b_{1.4} \\ 0 & 1 & 0 & b_{2.4} \\ 0 & 0 & 1 & b_{3.4} \end{pmatrix}$$

In this form the solutions may be read off immediately, as in Worked Example 9.9.

WORKED EXAMPLE 9.9

GAUSS–JORDAN ELIMINATION

Solve the following equations by Gauss–Jordan elimination:

$$2x + y + z = 12$$
$$6x + 5y - 3z = 6$$
$$4x - y + 3z = 5$$

Solution

Rearrange the equations to have variables on the LHS and constants on the RHS, as for Gaussian elimination. Then write them as an augmented matrix. Start by carrying out the Gaussian elimination, i.e. reducing the augmented matrix to upper triangular form. Since this is the same set of equations as those in Worked Example 9.8, we will continue from this point.

Action	Augmented matrix	Calculations
Make the tinted −8 into a 1 by dividing row 3^2 by −8	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & \mathbf{-8} & -64 \end{pmatrix}$ $\begin{matrix}(1)^1\\(2)^2\\(3)^2\end{matrix}$	
Add multiples of row 3 to rows 1 and 2 to generate 0s in column 3	$\begin{pmatrix} 1 & 0.5 & 0.5 & 6 \\ 0 & 1 & -3 & -15 \\ 0 & 0 & 1 & 8 \end{pmatrix}$ $\begin{matrix}(1)^1\\(2)^2\\(3)^3\end{matrix}$	
$(-0.5 \times \text{row } 3^3) + \text{row } 1^1$ $(3 \times \text{row } 3^3) + \text{row } 2^2$	$\begin{pmatrix} 1 & 0.5 & 0 & 2 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 8 \end{pmatrix}$ $\begin{matrix}(1)^2\\(2)^3\\(3)^3\end{matrix}$	Calculate $(-0.5 \times \text{row } 3^3)$ → (0 0 −0.5 −4) Calculate $(3 \times \text{row } 3^3)$ → (0 0 3 24)
Add multiples of row 2 to row 1 to generate 0s in column 2	$\begin{pmatrix} 1 & 0 & 0 & -2.5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 8 \end{pmatrix}$ $\begin{matrix}(1)^3\\(2)^3\\(3)^3\end{matrix}$	Calculate $(-0.5 \times \text{row } 2^3)$ → (0 −0.5 0 −4.5)

Write down the equations from the augmented matrix:

$$x + 0y + 0z = -2.5$$
$$0x + y + 0z = 9$$
$$0x + 0y + z = 8$$

or read off the solution directly: $x = -2.5$, $y = 9$, $z = 8$

Note: If the elimination process produces fewer equations than unknowns then there is no unique solution; see Chapter 8 of Hartley and Wynn-Evans.

Gauss–Jordan elimination will be used in Section 9.5 to calculate the inverse of a matrix.

Progress Exercises 9.3 Gaussian and Gauss–Jordan Elimination

1. Write the following system of equations as an augmented matrix:

$$3x + 3y + 6z = 12$$
$$x - 3y + 5z = 5$$
$$2x + 10y - 3z = 0$$

(a) Reduce the augmented matrix to upper triangular form, then solve by back substitution.
(b) Solve by Gauss–Jordan elimination.

2. Solve the following systems of equations by (i) Gaussian elimination and (ii) Gauss–Jordan elimination:

(a)
$$x + y = 12$$
$$2x + 5y + 2z = 20$$
$$6x + 3y + 6z = 0$$

(b)
$$x + y = 12$$
$$2x - 5y + 2z = 20$$
$$6x + 3y + 6z = 0$$

(c)
$$x + y = 12$$
$$2x + 2y = 20$$
$$6x + 3y + 6z = 0$$

(d)
$$x + y - 2z = 12$$
$$x - 5y + 4z = 20$$
$$-6x + 3y - 15z = 0$$

Solve the equations in questions 3 to 7 and leave your answers as fractions.

3.
$$3x + 4y - 9z = -2$$
$$6x + 15z - 21 = 0$$
$$5x - 4y - 9 = 0$$

4.
$$P_1 + 4P_2 + 8P_3 = 26$$
$$5P_1 + 7P_2 = 38$$
$$8P_1 + 12P_2 + 2P_3 = 66$$

5.
$$2Y - 5C + 0.8T = 580$$
$$-Y + C + 0.6T + 340 = 0$$
$$0.4Y - T = 100$$

8. Use an elimination method to find the equilibrium prices and quantities where the supply and demand functions for each good are as follows:

$$Q_{d1} = 50 - 2P_1 + 5P_2 - 3P_3 \qquad Q_{s1} = 8P_1 - 5$$
$$Q_{d2} = 22 + 7P_1 - 2P_2 + 5P_3 \qquad Q_{s2} = 12P_2 - 5$$
$$Q_{d3} = 17 + P_1 + 5P_2 - 3P_3 \qquad Q_{s3} = 4P_3 - 1$$

9.4 Determinants

At the end of this section you should be able to:

- Evaluate 2 × 2 and 3 × 3 determinants
- State and use Cramer's rule to solve two and three simultaneous equations in the same number of unknowns.
- Find equilibrium values for the national income model and other applications.

9.4.1 Evaluate 2 × 2 determinants

❒ Determinants: definitions

- A determinant is a square array of numbers or symbols, for example,

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad B = \begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} \qquad D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

- The dimensions of a determinant are stated as (number of rows, r) × (number of columns, c). Therefore, the dimensions of the determinants A, B and D are

$$A\text{: } 2 \times 2, \qquad B\text{: } 2 \times 2, \qquad D\text{: } 3 \times 3.$$

The elements within a determinant (or matrix) are referred to by the row and column in which the element occurs, for example,

$$\text{row 1, col. 1, etc.} \downarrow$$

$$B = \begin{vmatrix} b_{1.1} = 2 & b_{1.2} = 5 \\ b_{2.1} = 3 & b_{2.2} = -4 \end{vmatrix}$$

❒ How to evaluate a 2 × 2 determinant

A 2 × 2 determinant is evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a)(d) - (c)(b) \tag{9.20}$$

Hence the value of determinant B is:

$$\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = (2)(-4) - (3)(5) = -8 - (15) = -23$$

Warning: Most mistakes made in evaluating determinants arise from signs. So use brackets.

9.4.2 Use determinants to solve equations: Cramer's rule

You might wonder 'how can determinants be used to solve equations?' The use of determinants are demonstrated through Worked Example 9.10.

WORKED EXAMPLE 9.10

USING DETERMINANTS TO SOLVE SIMULTANEOUS EQUATIONS

Solve the simultaneous equations by eliminating one variable, then solve for the other variable.

(a) $\begin{aligned} 2x + 5y &= 10 \\ 3x + 4y &= 5.8 \end{aligned}$ (b) $\begin{aligned} a_1x + b_1y &= d_1 \\ a_2x + b_2y &= d_2 \end{aligned}$

Describe the method of solution in general terms using the equations given in (b). Hence, deduce a method of solution which uses determinants.

Solution

To describe the method of solution in general we shall solve (a) and (b) in parallel.

(a)

(1) $2x + 5y = 10$

(2) $3x + 4y = 5.8$

To eliminate y, multiply equation (1) by 4 and equation (2) by 5.

(1) × 4 $8x + 20y = 40$

(2) × 5 $15x + 20y = 29$

Subtract

$$8x - 15x + 0y = 40 - 29$$

Gathering the x terms

$$-7x = 11$$

Divide across by -7

$$x = \frac{11}{-7} = -1.57$$

(b)

(1) $a_1x + b_1y = d_1$

(2) $a_2x + b_2y = d_2$

To eliminate y, multiply equation (1) by b_2 and equation (2) by b_1

(1) × b_2 $a_1b_2x + b_1b_2y = d_1b_2$

(2) × b_1 $a_2b_1x + b_1b_2y = d_2b_1$

Subtract

$$a_1b_2x - a_2b_1x = d_1b_2 - d_2b_1$$

Gathering the x terms

$$(a_1b_2 - a_2b_1)x = d_1b_2 - d_2b_1$$

Divide across by $(a_1b_2 - a_2b_1)$

$$x = \frac{d_1b_2 - d_2b_1}{(a_1b_2 - a_2b_1)} \qquad \textbf{(9.21)}$$

Looking carefully at the numerator and the denominator of equation (9.21), notice that in each case there is (product of two values) – (product of two values). This is exactly the same format that was used when evaluating a 2 × 2 determinant. Therefore, the numerator and denominator in equation (9.21) may be

written as determinants:

$$x = \frac{d_1b_2 - d_2b_1}{(a_1b_2 - a_2b_1)} = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.22}$$

Similarly, if x was eliminated in equations (b) and solved for y, it would be found that:

$$y = \frac{a_1d_2 - a_2d_1}{(a_1b_2 - a_2b_1)} = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.23}$$

The formulae for x and y are now examined in detail. A general rule is deduced, called Cramer's rule, which uses determinants to solve simultaneous equations.

❒ Cramer's rule

The solution of the simultaneous equations

$$a_1x + b_1y = d_1$$
$$a_2x + b_2y = d_2$$

is given by the formulae

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

When these formulae are examined in detail, it is noted that:

- The denominator is the same in each:

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{9.24}$$

same

the denominator in each case is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

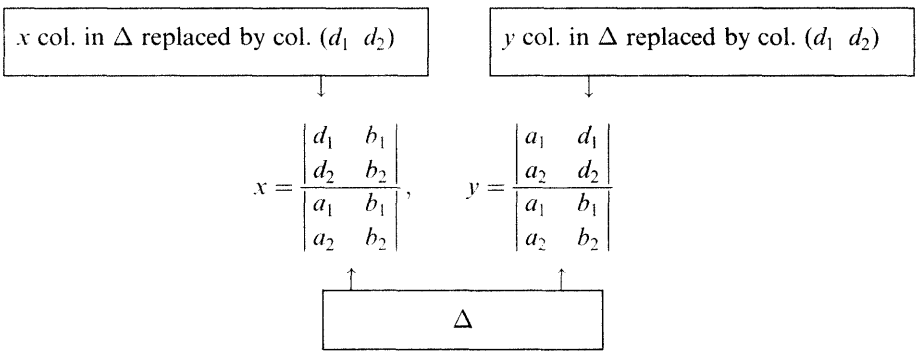
The columns in this determinant are related to the original equations as follows:

$$\begin{array}{l} a_1x + b_1y = d_1 \\ a_2x + b_2y = d_2 \\ \quad\downarrow \quad\quad \downarrow \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{array}$$

Column 1 consists of the coefficients of the x variables from the original set of equations. Column 2 consists of the coefficients of the y variables from the original set of equations. The determinant of the coefficients shall be referred to as Δ, that is,

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Looking again at equation (9.24), a rule for writing down the numerators may be established:



The determinant in which the column of x-coefficients are replaced by the column constants is referred to as Δ_x:

$$\Delta_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$$

The determinant in which the column of y-coefficients are replaced by the column constants is referred to as Δ_y:

$$\Delta_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

These observations are summarised:

The solution of two simultaneous equations in two unknowns is given by

$$x = \frac{\Delta_x}{\Delta} \qquad y = \frac{\Delta_y}{\Delta} \tag{9.25}$$

In fact, this general formula may be extended to any number of equations in the same number of unknowns. This general rule is called Cramer's rule. For example,

Cramer's rule
The solution of three linear equations in three unknowns is given by

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}, \qquad z = \frac{\Delta_z}{\Delta} \tag{9.26}$$

Note: if $\Delta = 0$, there is division by zero when applying Cramer's rule formulae. $\Delta = 0$ means that the set of equations has no unique solution (see Chapter 3).

WORKED EXAMPLE 9.11

USING CRAMER'S RULE TO SOLVE SIMULTANEOUS EQUATIONS

Use Cramer's rule to solve the equations:

$$\text{(a)} \quad \begin{aligned} y &= 10x + 12 \\ 4x + 2y &= 36 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} P &= 50 - 2Q \\ P &= 5 + 3Q \end{aligned}$$

Solution

In each case, Cramer's rule is used:

The solution of two simultaneous equations in two unknowns is

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}$$

The procedure is as follows:

Step 1. Write the equations in order: LHS (variable terms in the same order) = RHS (constants).
Step 2. Write down the determinants Δ, Δ_x, Δ_y. Evaluate these determinants.
Step 3. Use Cramer's rule to solve for the unknowns.

$$\text{(a)} \quad \begin{aligned} y &= 10x + 12 \\ 4x + 2y &= 36 \end{aligned}$$

Step 1

$$\begin{aligned} -10x + y &= 12 \\ 4x + 2y &= 36 \end{aligned}$$

Step 2

$$\Delta = \begin{vmatrix} -10 & 1 \\ 4 & 2 \end{vmatrix} = (-10)(2) - (4)(1)$$
$$= -20 - 4 = -24$$

$$\Delta_x = \begin{vmatrix} 12 & 1 \\ 36 & 2 \end{vmatrix} = (12)(2) - (36)(1)$$
$$= 24 - 36 = -12$$

$$\text{(b)} \quad \begin{aligned} P &= 50 - 2Q \\ P &= 5 + 3Q \end{aligned}$$

Step 1

$$\begin{aligned} P + 2Q &= 50 \\ P - 3Q &= 5 \end{aligned}$$

Step 2

$$\Delta = \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = (1)(-3) - (1)(2)$$
$$= -3 - 2 = -5$$

$$\Delta_P = \begin{vmatrix} 50 & 2 \\ 5 & -3 \end{vmatrix} = (50)(-3)$$
$$- (5)(2) = -160$$

$$\Delta_y = \begin{vmatrix} -10 & 12 \\ 4 & 36 \end{vmatrix} = (-10)(36) - (4)(12) = -408$$

Step 3. Use Cramer's rule.

$$x = \frac{\Delta_x}{\Delta} = \frac{-12}{-24} = 0.5$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-408}{-24} = 17$$

$$\Delta_Q = \begin{vmatrix} 1 & 50 \\ 1 & 5 \end{vmatrix} = (1)(5) - (1)(50) = -45$$

Step 3. Use Cramer's rule.

$$P = \frac{\Delta_P}{\Delta} = \frac{-160}{-5} = 32$$

$$Q = \frac{\Delta_Q}{\Delta} = \frac{-45}{-5} = 9$$

WORKED EXAMPLE 9.12

FIND THE MARKET EQUILIBRIUM USING CRAMER'S RULE

Given the supply and demand functions for two related goods, A and B,

Good A: $\{Q_{da} = 30 - 8P_a + 2P_b$, $\{Q_{sa} = -15 + 7P_a$

Good B: $\{Q_{db} = 28 + 4P_a - 6P_b$, $\{Q_{sb} = 12 + 2P_b$

(a) Write down the equilibrium condition for each good. Hence, deduce two equations in P_a and P_b.
(b) Use Cramer's rule to find the equilibrium prices and quantities for goods A and B.

Solution

(a) The equilibrium condition for each good is that $Q_s = Q_d$.

For good A:

$$-15 + 7P_a = 30 - 8P_a + 2P_b$$
$$15P_a - 2P_b = 45$$

For good B:

$$12 + 2P_b = 28 + 4P_a - 6P_b$$
$$-4P_a + 8P_b = 16$$

Therefore, the simultaneous equations are:

(1) $15P_a - 2P_b = 45$

(2) $-4P_a + 8P_b = 16$

(b) Applying Cramer's rule:

$$P_a = \frac{\Delta_{P_a}}{\Delta} = \frac{\begin{vmatrix} 45 & -2 \\ 16 & 8 \end{vmatrix}}{\begin{vmatrix} 15 & -2 \\ -4 & 8 \end{vmatrix}} = \frac{(45)(8) - (16)(-2)}{(15)(8) - (-4)(-2)} = \frac{360 - (-32)}{120 - (8)} = \frac{392}{112} = 3.5$$

$$P_b = \frac{\Delta_{P_b}}{\Delta} = \frac{\begin{vmatrix} 15 & 45 \\ -4 & 16 \end{vmatrix}}{\begin{vmatrix} 15 & -2 \\ -4 & 8 \end{vmatrix}} = \frac{(15)(16) - (-4)(45)}{(15)(8) - (-4)(-2)} = \frac{240 - (-180)}{120 - (8)} = \frac{420}{112} = 3.75$$

Substituting these values of P_a and P_b into any of the original equations, solve for Q_a and Q_b. Solution: $Q_a = 9.5$, $Q_b = 19.5$.

❒ General expressions for equilibrium in the income-determination model

Cramer's rule may be used to find general expressions for the equilibrium level of income, consumption, investment, etc. in the income determination model.

WORKED EXAMPLE 9.13

USE CRAMER'S RULE FOR THE INCOME-DETERMINATION MODEL

Given the general income-determination model (no government sector):

$$Y = C + I \tag{9.27}$$

$$C = C_0 + bY \tag{9.28}$$

where $I = I_0$, $0 < b < 1$ (b, I_0 and C_0 are constants).

(a) Write equations (9.24) and (9.25) in the form:

$$a_1 Y + a_2 C = a_3 \qquad a_1, a_2, a_3 \text{ are constants}$$

(b) Hence, use Cramer's rule to express the equilibrium levels of income (Y_e) and consumption (C_e) in terms of the constants b, I_0 and C_0.

Solution

(a) Rearranging equations (9.27) and (9.28) in the required form, variable terms on LHS and constant terms on RHS, and substituting I_0 for I:

$$Y - C = I_0 \tag{9.29}$$

$$-bY + C = C_0 \tag{9.30}$$

(b) Solve equations (9.29) and (9.30) for Y and C by Cramer's rule:

$$Y_e = \frac{\Delta_Y}{\Delta} = \frac{I_0 + C_0}{1 - b} \qquad \text{and} \qquad C_e = \frac{\Delta_C}{\Delta} = \frac{C_0 + bI_0}{1 - b}$$

Since:

$$\Delta = \begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = 1 - (-b)(-1) = 1 - b$$

$$\Delta_Y = \begin{vmatrix} I_0 & -1 \\ C_0 & 1 \end{vmatrix} = (I_0) - (-C_0) = I_0 + C_0$$

$$\Delta_C = \begin{vmatrix} 1 & I_0 \\ -b & C_0 \end{vmatrix} = C_0 - (-b)(I_0) = C_0 + bI_0$$

Progress Exercises 9.4 2 × 2 Determinants, with Applications

1. Evaluate the following determinants:

(a) $\begin{vmatrix} 4 & 1 \\ 9 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -2 & 0 \\ 2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 4 & -3 \\ 1 & 8 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix}$ (e) $\begin{vmatrix} 6 & -1 \\ -6 & 2 \end{vmatrix}$

(f) $\begin{vmatrix} 6 & -1 \\ -6 & 1 \end{vmatrix}$ (g) $\begin{vmatrix} c & 2c \\ 1 & c \end{vmatrix}$ (h) $\begin{vmatrix} a & (1-a) \\ -a & a \end{vmatrix}$ (i) $\begin{vmatrix} \frac{1}{a} & -\frac{1}{1-a} \\ -a & a \end{vmatrix}$

Use Cramer's rule to solve the simultaneous equations (correct to two decimal places) in questions 2 to 7.

2. $x + 2y = 10, \quad 5x + 8y = 40$

3. $2x - y = 12, \quad 21x + 12y = 63$

4. $5P_1 + 9P_2 = 36, \quad 12P_1 + 2P_2 = 16$

5. $P = 100 - 8Q, \quad P = 30 + 5Q$

6. $50P + 8Q - 190 = 0, \quad P = 25 + 7Q$

7. $Y = 0.2r + 20, \quad Y = -0.05r + 42$

8. The demand and supply functions for two related products (1: pens, 2: paper) are given by the equations:

$$Q_{d1} = 30 - P_1 + 4P_2, \quad Q_{s1} = 3P_1 - 6$$
$$Q_{d2} = 36 + 3P_1 - 2P_2, \quad Q_{s2} = 12P_2 - 4$$

Use Cramer's rule to find the equilibrium prices and quantities.

9. Given the general income determination model:

(1) $Y = C + I_0$ and (2) $C = C_0 + bY$ where $0 < b < 1$, I_0 and C_0 are constants,

(a) Write equations (1) and (2) in the form $a_1 Y + a_2 C = a_3$ where a_1, a_2, a_3 are constants.
(b) Hence use Cramer's rule to express the equilibrium condition for income (Y) and consumption (C) in terms of the constants b, I_0 and C_0.

10. If the equilibrium condition in
(i) the goods market is given by the equation

$Y = C + I$ where $C = 237.8 + 0.2Y$ and $I = 10 - 0.4r$

(ii) the money market is given by the equation

$M_d = M_s$ where $M_d = 100 + 0.1Y - 0.3r$ and $M_s = 129.225$

(a) Write the equilibrium equations for each market in the form $aY + cr = c$ where a, b and c are constants
(b) Use Cramer's rule to solve for the equilibrium levels of income (Y) and interest rate (r), for which the product and money markets are simultaneously in equilibrium.

11. Given the demand function for two goods,

$$Q_1 = 5 - P_1 + 4P_2$$
$$Q_2 = -3P_2 + 2P_1$$

use Cramer's rule to derive expressions for P_1 and P_2 in terms of Q_1 and Q_2.

9.4.3 Evaluate 3 × 3 determinants

Using Cramer's rule to solve three equations in three unknowns involves the evaluation of 3 × 3 determinants. This is quite a different procedure from evaluating 2 × 2 determinants. To evaluate the general 3 × 3 determinant A, proceed as follows:

$$A = \begin{vmatrix} a_{1.1} & a_{1.2} & a_{1.3} \\ a_{2.1} & a_{2.2} & a_{2.3} \\ a_{3.1} & a_{3.2} & a_{3.3} \end{vmatrix} = (a_{1.1})\begin{vmatrix} a_{2.2} & a_{2.3} \\ a_{3.2} & a_{3.3} \end{vmatrix} - (a_{1.2})\begin{vmatrix} a_{2.1} & a_{2.3} \\ a_{3.1} & a_{3.3} \end{vmatrix} + (a_{1.3})\begin{vmatrix} a_{2.1} & a_{2.2} \\ a_{3.1} & a_{3.2} \end{vmatrix} \tag{9.31}$$

Note: brackets are used to avoid errors when substituting negative numbers.

WORKED EXAMPLE 9.14

EVALUATION OF A 3 × 3 DETERMINANT

Evaluate the 3 × 3 determinant, D, where

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

Solution

Applying the method given in equation (9.31), the value of determinant D is calculated as:

$$\begin{aligned} D &= 1\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} - (0)\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 1[(2)(2) - (3)(3)] - 0[(2)(2) - (1)(3)] + (-2)[(2)(3) - (1)(2)] \\ &= 1(-5) + (0)(1) + (-2)(4) = -5 + 0 - 8 = -13 \end{aligned}$$

WORKED EXAMPLE 9.15

SOLVE THREE SIMULTANEOUS EQUATIONS BY CRAMER'S RULE

The equilibrium condition for three related products simplifies to the following equations:

$$\begin{aligned} 15P_1 - 4P_2 - 7P_3 &= 14 \\ -4P_1 + 6P_2 - 2P_3 &= 34 \\ -3P_1 - 2P_2 + 12P_2 &= 1 \end{aligned}$$

Use Cramer's rule to solve for the equilibrium prices P_1, P_2 and P_3.

Solution

Go through the usual steps for Cramer's rule:

Step 1. The equations are already arranged in the required form

LHS (variable terms, in order) = RHS (constants).

Step 2. Write out and evaluate the four determinants: Δ, Δ_{P_1}, Δ_{P_2}, Δ_{P_3}.

The determinants are:

$$\Delta = \begin{vmatrix} 15 & -4 & -7 \\ -4 & 6 & -2 \\ -3 & -2 & 12 \end{vmatrix} \quad \Delta_{P_1} = \begin{vmatrix} 14 & -4 & -7 \\ 34 & 6 & -2 \\ 1 & -2 & 12 \end{vmatrix} \quad \Delta_{P_2} = \begin{vmatrix} 15 & 14 & -7 \\ -4 & 34 & -2 \\ -3 & 1 & 12 \end{vmatrix}$$

$$\Delta_{P_3} = \begin{vmatrix} 15 & -4 & 14 \\ -4 & 6 & 34 \\ -3 & -2 & 1 \end{vmatrix}$$

Evaluate each determinant as follows:

$$\Delta = \begin{vmatrix} 15 & -4 & -7 \\ -4 & 6 & -2 \\ -3 & -2 & 12 \end{vmatrix} = 15\begin{vmatrix} 6 & -2 \\ -2 & 12 \end{vmatrix} - (-4)\begin{vmatrix} -4 & -2 \\ -3 & 12 \end{vmatrix} + (-7)\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$$

$$= 15[(6)(12) - (-2)(-2)] + 4[(-4)(12) - (-3)(-2)] - (7)[(-4)(-2) - (-3)(6)]$$

$$= (15)(68) + (4)(-54) + (-7)(26)$$

$$= 1020 - 216 - 182 = 622$$

$$\Delta_{P_1} = \begin{vmatrix} 14 & -4 & -7 \\ 34 & 6 & -2 \\ 1 & -2 & 12 \end{vmatrix} = 14\begin{vmatrix} 6 & -2 \\ -2 & 12 \end{vmatrix} - (-4)\begin{vmatrix} 34 & -2 \\ 1 & 12 \end{vmatrix} + (-7)\begin{vmatrix} 34 & 6 \\ 1 & -2 \end{vmatrix}$$

$$= 14[(6)(12) - (-2)(-2)] + 4[(34)(12) - (1)(-2)] - 7[(34)(-2) - (1)(6)]$$

$$= 14[68] + 4[410] - 7[-74]$$

$$= 952 + 1640 + 518 = 3110$$

Similarly, Δ_{P_2} is evaluated as follows:

$$\Delta_{P_2} = \begin{vmatrix} 15 & 14 & -7 \\ -4 & 34 & -2 \\ -3 & 1 & 12 \end{vmatrix} = 15\begin{vmatrix} 34 & -2 \\ 1 & 12 \end{vmatrix} - 14\begin{vmatrix} -4 & -2 \\ -3 & 12 \end{vmatrix} + (-7)\begin{vmatrix} -4 & 34 \\ -3 & 1 \end{vmatrix}$$

Now evaluate each 2×2 determinant:

$$\Delta_{P_2} = 15[(34)(12) - (1)(-2)] - 14[(-4)(12) - (-3)(-2)] - 7[(-4)(1) - (-3)(34)]$$

$$= 15[410] - 14[-54] - 7[98]$$

$$= 6150 + 756 - 686 = 6220$$

Similarly, Δ_{P_3} is evaluated as

$$\Delta_{P_3} = \begin{vmatrix} 15 & -4 & 14 \\ -4 & 6 & 34 \\ -3 & -2 & 1 \end{vmatrix} = 15\begin{vmatrix} 6 & 34 \\ -2 & 1 \end{vmatrix} - (-4)\begin{vmatrix} -4 & 34 \\ -3 & 1 \end{vmatrix} + (14)\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$$

Evaluate the 2×2 determinants,

$$\begin{aligned} \Delta_{P_3} &= 15[(6)(1) - (-2)(34)] + 4[(-4)(1) - (-3)(34)] \\ &\quad + 14[(-4)(-2) - (-3)(6)] \\ &= 15[74] + 4[98] + 14[26] \\ &= 1110 + 392 + 364 = 1866 \end{aligned}$$

Step 3. Use Cramer's rule to solve for P_1, P_2 and P_3:

$$P_1 = \frac{\Delta_{P_1}}{\Delta} = \frac{3110}{622} = 5, \quad P_2 = \frac{\Delta_{P_2}}{\Delta} = \frac{6220}{622} = 10, \quad P_3 = \frac{\Delta_{P_3}}{\Delta} = \frac{1866}{622} = 3$$

❒ Applications

Cramer's rule may be used in any application which involves the solution of three or more simultaneous linear equations, such as in equilibrium, break-even, etc. In Worked Example 9.12 the equilibrium for a two-product market was calculated. The equilibrium for markets with three products or more may now be calculated. Worked Example 9.13, the income-determination model, may also be extended to three or more variables.

WORKED EXAMPLE 9.16

EQUILIBRIUM LEVELS IN THE NATIONAL INCOME MODEL

Given the following national income model:

$$Y = C + I_0 + G_0$$

$$C = C_0 + b(Y - T), \quad \text{and} \quad T = T_0 + tY$$

where Y = income, C = consumption, T = taxation, C_0, b, t, I_0, G_0 and T_0 (autonomous taxation) are constants and $C_0 > 0$; $0 < b < 1$; $0 < t < 1$:

(a) Write this model as three equations in terms of the variables Y, C and T.
(b) Use Cramer's rule to derive expressions for the equilibrium level of income, consumption and taxation.

Solution

(a) The variable terms containing Y, C and T are arranged on the LHS of each equation, the constants on the RHS:

$$\begin{aligned} Y - C &= I_0 + G_0 \\ -bY + C + bT &= C_0 \\ -tY + T &= T_0 \end{aligned}$$

(b) Using Cramer's rule, solve for Y, C, T:

$$Y = \frac{\Delta_Y}{\Delta} = \frac{I_0 + G_0 + C_0 - bT_0}{1 - b + bt} = \text{the equilibrium level of income}$$

$$C = \frac{\Delta_C}{\Delta} = \frac{C_0 - bT_0 + b(1-t)(I_0 + G_0)}{1 - b + bt} = \text{equilibrium level of consumption}$$

$$T = \frac{\Delta_T}{\Delta} = \frac{T_0(1-b) + t(C_0 + I_0 + G_0)}{1 - b + bt} = \text{equilibrium level of taxation}$$

Since:

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = (1)\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + (0) = 1 + (-b - (-bt))$$

$$= 1 - b + bt$$

$$\Delta_Y = \begin{vmatrix} I_0 + G_0 & -1 & 0 \\ C_0 & 1 & b \\ T_0 & 0 & 1 \end{vmatrix} = (I_0 + G_0)\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} C_0 & b \\ T_0 & 1 \end{vmatrix} + (0)$$

$$= (I_0 + G_0) + (C_0 - bT_0)$$

$$\Delta_C = \begin{vmatrix} 1 & I_0 + G_0 & 0 \\ -b & C_0 & b \\ -t & T_0 & 1 \end{vmatrix} = (1)\begin{vmatrix} C_0 & b \\ T_0 & 1 \end{vmatrix} - (I_0 + G_0)\begin{vmatrix} -b & b \\ -t & 1 \end{vmatrix} + (0)$$

$$= (C_0 - bT_0) - (I_0 + G_0)(-b - (-bt))$$

$$= C_0 - bT_0 + b(I_0 + G_0)(1 - t)$$

$$\Delta_T = \begin{vmatrix} 1 & -1 & I_0 + G_0 \\ -b & 1 & C_0 \\ -t & 0 & T_0 \end{vmatrix} = (1)\begin{vmatrix} 1 & C_0 \\ 0 & T_0 \end{vmatrix} - (-1)\begin{vmatrix} -b & C_0 \\ -t & T_0 \end{vmatrix}$$

$$+ (I_0 + G_0)\begin{vmatrix} -b & 1 \\ -t & 0 \end{vmatrix}$$

$$= T_0 - bT_0 + tC_0 + (I_0 + G_0)(0 - (-t))$$

$$= T_0(1 - b) + t(C_0 + I_0 + G_0)$$

Progress Exercises 9.5 *3 × 3 Determinants, with Applications*

1. Evaluate each of the following determinants:

$$\text{(a)}\ \begin{vmatrix} 1 & 2 & -5 \\ 0 & 6 & 5 \\ -1 & 2 & 7 \end{vmatrix} \qquad \text{(b)}\ \begin{vmatrix} -3 & 0 & 3 \\ 3 & 2 & 6 \\ 4 & 0 & 9 \end{vmatrix} \qquad \text{(c)}\ \begin{vmatrix} -2 & 3 & 2 \\ 5 & 12 & 2 \\ 3 & -4.5 & 3 \end{vmatrix}$$

2. Solve the following system of equations by Cramer's rule

$$\text{(a)}\quad \begin{aligned} x + y &= 12 \\ 2x + 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} x + y &= 12 \\ 2x - 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$$

$$\text{(c)}\quad \begin{aligned} x + y &= 12 \\ 2x + 2y &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned} \qquad \text{(d)}\quad \begin{aligned} x + y - 2z &= 12 \\ x - 5y + 4z &= 20 \\ -6x + 3y - 15z &= 0 \end{aligned}$$

Use Cramer's rule to solve the following equations in questions 3 to 7.

3.
$$\begin{aligned} 3x + 4y - 9z &= -2 \\ 6x + 15z - 21 &= 0 \\ 5x + 4y - 9 &= 0 \end{aligned}$$

4.
$$\begin{aligned} P_1 + 4P_2 + 8P_3 &= 26 \\ 5P_1 + 7P_2 &= 38 \\ 8P_1 + 12P_2 + 2P_3 &= 66 \end{aligned}$$

5.
$$\begin{aligned} 2Y - 5C + 0.8T &= 580 \\ -Y + C + 0.6T + 340 &= 0 \\ 0.4Y - T &= 100 \end{aligned}$$

6.
$$\begin{aligned} 5x + 5y - 4z &= 15 \\ 7x - 3y + 4z &= 15 \\ z &= 2x - 8y \end{aligned}$$

7.
$$\begin{aligned} 5x + 1.7z &= 21.6 \\ y - z &= 0 \\ 2.5y - 2.5x &= 1.35x - 10.8 \end{aligned}$$

8. Use Cramer's rule to find the equilibrium prices and quantities where the supply and demand functions for each good are

$$\begin{aligned} Q_{d1} &= 50 - 2P_1 + 5P_2 - 3P_3, & Q_{s1} &= 8P_1 - 5 \\ Q_{d2} &= 22 + 7P_1 - 2P_2 + 5P_3, & Q_{s2} &= 12P_2 - 5 \\ Q_{d3} &= 17 + P_1 + 5P_2 - 3P_3, & Q_{s3} &= 4P_3 - 1 \end{aligned}$$

9. (a) State Cramer's rule.
Show graphically how a system of two equations in two unknowns has:

(i) no solution, (ii) a unique solution, (iii) infinitely many solutions.

Is it possible to solve all three systems above by Cramer's rule?
(b) Use Cramer's rule to solve for P_1, P_2 and P_3.

$$\begin{aligned} 0.5P_1 + 11.6P_2 - 8P_3 &= -47.39 \\ 1.9P_2 + 4.5P_3 &= 59.94 \\ 0.8P_1 + 3.5P_3 &= 49.7 \end{aligned}$$

9.5 The Inverse Matrix and Input/Output Analysis

At the end of this section you should be able to:

- Calculate the inverse of a matrix
- Use the inverse matrix to solve equations
- Solve problems related to input/output analysis

Section 9.4 used determinants to solve three equations in three unknowns. This section uses the inverse matrix for the same purpose. It also introduces one other application of the inverse matrix, input/output analysis.

❒ The inverse matrix

Section 9.2 introduced matrix arithmetic: addition, subtraction and multiplication. For each arithmetic operation, the restrictions imposed were noted because matrices were used instead of single numbers. In matrix multiplication the order of multiplication generally produced different answers. When a matrix B is divided by a matrix A, B is multiplied by A^{-1}, the inverse of A, but there are two possibilities:

$$\frac{B}{A} = \begin{cases} BA^{-1} \\ A^{-1}B \end{cases}$$

each of which will generally give a different answer. Therefore matrix division is not used directly; instead we multiply by the inverse of the matrix.

9.5.1 To find the inverse of a matrix: elimination method

To find the inverse of square matrix, set up the augmented matrix, consisting of the matrix whose inverse is required, and a unit matrix of the same dimension.

WORKED EXAMPLE 9.17

THE INVERSE OF A MATRIX: ELIMINATION METHOD

(a) Find the inverse of the matrix

$$D = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

by Gauss–Jordan elimination.
(b) Show that $DD^{-1} = I$.

Solution

Write out the augmented matrix consisting of the matrix D and the unit matrix of the same dimension:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

Carry out Gauss–Jordan elimination on the matrix D by the method given in Worked Example 9.9.

Action	Augmented matrix	Calculations
	$\left(\begin{array}{ccc\|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right)\begin{matrix}(1)\\(2)\\(3)\end{matrix}$	
row $2 + (-2 \times$ row $1)$ row $3 + (-1 \times$ row $1)$	$\left(\begin{array}{ccc\|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 7 & -2 & 1 & 0 \\ 0 & 3 & 4 & -1 & 0 & 1 \end{array}\right)\begin{matrix}(1)\\(2)^1\\(3)^1\end{matrix}$	Calculate $(-2 \times$ row $1)$ $\rightarrow (-2 \quad 0 \quad 4 \quad -2 \quad 0 \quad 0)$ Calculate $(-1 \times$ row $1)$ $\rightarrow (-1 \quad 0 \quad 2 \quad -1 \quad 0 \quad 0)$
row $2^1 \times \frac{1}{2}$	$\left(\begin{array}{ccc\|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 4 & 0 & 0 & 1 \end{array}\right)\begin{matrix}(1)\\(2)^2\\(3)^1\end{matrix}$	
row $3^1 + (-3 \times$ row $2^2)$	$\left(\begin{array}{ccc\|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{13}{2} & 2 & -\frac{3}{2} & 1 \end{array}\right)\begin{matrix}(1)\\(2)^2\\(3)^2\end{matrix}$	Calculate $(-3 \times$ row $2^2)$ $\rightarrow (0 \quad -3 \quad -\frac{21}{2} \quad 3 \quad -\frac{3}{2} \quad 0)$
row $3^2 \times -\frac{2}{13}$	$\left(\begin{array}{ccc\|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{13} & \frac{3}{13} & -\frac{2}{13} \end{array}\right)\begin{matrix}(1)\\(2)^2\\(3)^3\end{matrix}$	
row $1 + (2 \times$ row $3^3)$ row $2^2 + (-\frac{7}{2} \times$ row $3^3)$	$\left(\begin{array}{ccc\|ccc} 1 & 0 & 0 & \frac{5}{13} & \frac{6}{13} & \frac{4}{13} \\ 0 & 1 & 0 & \frac{1}{13} & -\frac{4}{13} & \frac{7}{13} \\ 0 & 0 & 1 & -\frac{4}{13} & \frac{3}{13} & -\frac{2}{13} \end{array}\right)\begin{matrix}(1)^1\\(2)^3\\(3)^3\end{matrix}$	Calculate $(2 \times$ row $3^3)$ $\rightarrow (0 \quad 0 \quad 2 \quad -\frac{8}{13} \quad \frac{6}{13} \quad -\frac{4}{13})$ Calculate $(-\frac{7}{2} \times$ row $3^3)$ $\rightarrow (0 \quad 0 \quad -\frac{7}{2} \quad \frac{14}{13} \quad -\frac{10.5}{13} \quad \frac{7}{13})$

The original matrix, D, is now reduced to the unit matrix. The inverse of D is given by the transformed unit matrix: the 4th, 5th and 6th columns of the augmented matrix.

$$D^{-1} = \begin{pmatrix} \frac{5}{13} & \frac{6}{13} & -\frac{4}{13} \\ \frac{1}{13} & -\frac{4}{13} & \frac{7}{13} \\ -\frac{4}{13} & \frac{3}{13} & -\frac{2}{13} \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 6 & -4 \\ 1 & -4 & 7 \\ -4 & 3 & -2 \end{pmatrix}$$

Note: If division by zero arises at any stage, then D does not have an inverse.
(b) Every element in D^{-1} is multiplied by $1/13$. Before multiplying D^{-1} by D, factor out this scalar to simplify the arithmetic involved:

$$DD^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \frac{1}{13} \begin{pmatrix} 5 & 6 & -4 \\ 1 & -4 & 7 \\ -4 & 3 & -2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.5.2 To find the inverse of a matrix: cofactor method

Initially, some of the terminology associated with determinants and matrices is explained. The **minor** of an element is the determinant of what is left, when the row and column containing that element are crossed out. For example, in determinant

$$D = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

the minor of the first element in the first row, $d_{1,1}$, is

$$D = \begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-2} \\ \mathbf{2} & 2 & 3 \\ \mathbf{1} & 3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = (2)(2) - (3)(3) = -5$$

The **cofactor** of a given element is the minor of that element multiplied by either $+1$ or -1. If the given element is in the same position as $+1$ in the determinant of ± 1, given below, multiply the minor by $+1$, otherwise multiply the minor by -1. Here is the determinant of ± 1:

$$\begin{vmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{vmatrix}$$

The cofactor of an element is given the symbol C subscripted with the location (row, column) of that element. For example, $C_{1,1}$ is the cofactor of $d_{1,1}$ in determinant D above; it is calculated as follows:

$$C_{1,1} = (\text{minor of } d_{1,1}) \times (+1) = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \times (1) = (-5)(1) = -5$$

The minor, -5, was evaluated above.

The value of a 3×3 determinant is calculated as follows:

$$|D| = (d_{1,1} \times C_{1,1}) + (d_{1,2} \times C_{1,2}) + (d_{1,3} \times C_{1,3}) \qquad \textbf{(9.32)}$$

that is, the value of $|D|$ is the sum of the products of each element in row 1 and its **cofactor**. (In fact, $|D|$ may also be evaluated by summing the products of each element × cofactor from **any** one row or column. This is particularly useful if a row or column contains several zeros.) This method of evaluation is called Laplace expansion or the cofactor method.

Given a matrix A, the inverse of A is defined as follows:

$$A^{-1} = \frac{(C)^{\mathrm{T}}}{|A|}$$

where C^{T} is the matrix in which every element is replaced by its cofactor:

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{1.1} & C_{1.2} & C_{1.3} \\ C_{2.1} & C_{2.2} & C_{2.3} \\ C_{3.1} & C_{3.2} & C_{3.3} \end{pmatrix}^{\mathrm{T}} \tag{9.33}$$

Note: If

$$|A| = 0, \quad \frac{1}{|A|} \quad \rightarrow \quad \frac{1}{0},$$

A **has no inverse**, since division by zero is not defined. Therefore, the inverse of A is determined by:

Step 1. Evaluating $|A|$. If $|A| = 0$, there is no inverse.
Step 2. Calculating the cofactor of each element.
Step 3. Replacing each element by its cofactor, then transposing the matrix of cofactors (this matrix is called the adjoint of A).
Step 4. Multiplying the transposed matrix of cofactors by $1/|A|$.

❐ The inverse of a 2 × 2 matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $(ad - bc \neq 0)$

The proof is left as an exercise.

WORKED EXAMPLE 9.18

THE INVERSE OF A 3 × 3 MATRIX

(a) Find the inverse of the matrix

$$D = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

(b) Show that $DD^{-1} = I$.

Solution

(a) Use the definition of the inverse given in equation (9.30) to determine the inverse:

Step 1. Evaluate $|D|$. This is calculated by multiplying each element in row 1 by its cofactor. If the table for calculating cofactors is set up, the required cofactors are the first three cofactors in the table. So step 1 is deferred to step 2.

Step 2. Set up the table to calculate all the cofactors.

Element (by rows)	Minor × (sign) = cofactor
1	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}(1) = 4 - (9) = -5$
0	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}(-1) = [4 - (3)](-1) = -1$
−2	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}(1) = [6 - (2)](1) = 4$
	$\|D\|$ by row 1 $= (1)(-5) + 0(-1) + (-2)(4) = -13$
2	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 0 & -2 \\ 3 & 2 \end{vmatrix}(-1) = [0 - (-6)](-1) = -6$
2	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}(1) = [2 - (-2)](1) = 4$
3	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}(-1) = [3 - (0)](-1) = -3$
1	$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 0 & -2 \\ 2 & 3 \end{vmatrix}(1) = [0 - (-4)](1) = 4$

3 $$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}(-1) = [3-(-4)](-1) = -7$$

2 $$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}(1) = [2-(0)](1) = 2$$

Step 3. Replace each element in D by its cofactor. Then transpose.

$$C^{\mathrm{T}} = \begin{pmatrix} -5 & -1 & 4 \\ -6 & 4 & -3 \\ 4 & -7 & 2 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix} = (\text{adjoint of } D)$$

Step 4. Multiply the adjoint matrix by

$$\frac{1}{|D|} = \frac{1}{-13}$$

$$D^{-1} = \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$$

Every element in this matrix could be multiplied by $-(1/13)$, but this will introduce awkward fractions, so the scalar multiplication is usually left until a final single matrix is required.

(b) $$DD^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \times \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$$

$$= -\frac{1}{13}\begin{pmatrix} -13 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

❐ To write a system of equations in matrix form

In general, a system of equations, all written in the same format, may be expressed concisely in terms of matrices as follows:

$$\begin{array}{ll} (1) & a_1x + b_1y + c_1z = d_1 \\ (2) & a_2x + b_2y + c_2z = d_2 \\ (3) & a_3x + b_3y + c_3z = d_3 \end{array} \rightarrow \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \qquad (9.34)$$

This statement is written concisely as $AX = B$ (9.35)

Dimension $(3 \times 3)(3 \times 1) = (3 \times 1)$

If the matrix A is multiplied by the column matrix X, the result is a 3×1 matrix, in which each element is the LHS of equations (1), (2) and (3) above:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Equating the corresponding elements of these 3×1 matrices, AX and B, reproduces equations (1), (2) and (3).

Note: the columns of matrix A consist of the coefficients of x, y, z from equations (1), (2) and (3).

❒ To solve a set of equations using the inverse matrix

Premultiply both sides of equation (9.35), $AX = B$, by the inverse of matrix A:

$A^{-1}AX = A^{-1}B$ but $AA^{-1} = I$, the unit matrix

$IX = A^{-1}B$ but $IX = X$, since I in matrix multiplication behaves like a 1 in ordinary multiplication, therefore

$$X = A^{-1}B \tag{9.36}$$

where X is the column of unknowns, A is the matrix of coefficients, B is the column of constants from the RHS of the equations, taken in order. So, if the column matrix B is premultiplied by the inverse of the matrix A, the resulting column matrix is the solution for x, y, z.

WORKED EXAMPLE 9.19

SOLVE A SYSTEM OF EQUATIONS BY THE INVERSE MATRIX

Use the inverse matrix to solve the following simultaneous equations:

$$x - 2z = 4$$
$$2x + 2y + 3z = 15$$
$$x + 3y + 2z = 12$$

Solution

Step 1. Write all the equations in the same order: variables (in order) = RHS. Write down the matrices A and B. The matrix A is the matrix of coefficients of the three equations, all arranged in the same format; B is the column matrix consisting of the constants from the RHS of the equations.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 \\ 15 \\ 12 \end{pmatrix}$$

Step 2. Since $X = A^{-1}B$, determine the inverse of A. However, if you look carefully at A, you will see that this matrix is identical to the matrix D in Worked

Examples 9.17 and 9.18 in which the inverse of D was determined; therefore

$$A^{-1} = \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}$$

So proceed straight away to the next step.

Step 3. Premultiply the matrix B by the inverse of A:

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-13}\begin{pmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 15 \\ 12 \end{pmatrix} = -\frac{1}{13}\begin{pmatrix} -62 \\ -28 \\ -5 \end{pmatrix} = \begin{pmatrix} 4.77 \\ 2.15 \\ 0.38 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4.77 \\ 2.15 \\ 0.38 \end{pmatrix}$$

The solutions may simply be read off (correct to two decimal places), $x = 4.77$, $y = 2.15$, $z = 0.38$.

❒ Input/output analysis

Consider a three-sector economy, such as industry, agriculture and financial services. The output from any sector, such as agriculture, may be required by:

(i) The other sectors
(ii) The same sector
(iii) External demands, such as sales, exports, etc.

For example, the output from all three sectors (£million) in this three-sector economy could be distributed as follows:

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300

So the total output required from each sector must satisfy the final demand (sales, exports, etc.), as well as the demands from other sectors which require this as basic raw material or input. For example, the output from the agriculture sector is required as raw material by agriculture itself (calves, foodstuffs, etc.); industry requires agricultural output for processing; government and other agencies provide financial and other services. Finally, consumers and other agencies want to buy and export agricultural products.

The table above is perfectly balanced, but suppose the 'other demands' from each sector change. How is the total output required recalculated? The answer is to use matrices. One other fundamental assumption is needed before setting up our method, and that is:

In input/output analysis, **total input = total output** for each sector.

With this assumption, the input/output table can be completed, the total input to each sector is written in the last row. You will notice that the sum of individual inputs do not add up to the total input. Therefore a further row is added to the table to allow for other inputs.

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300
	Other inputs	70	425	5		
	Total input	600	900	300		

Divide the input to each sector by the total input to calculate the fraction (or, if you prefer, multiply the fraction by 100 and quote this as the percentage) of the total input which comes from all sectors:

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150/600	225/900	125/300	100	600
	Industry →	210/600	250/900	140/300	300	900
	Services →	170/600	0/900	30/300	100	300
	Total input	600/600	900/900	300/300		

The relationship between the total output from all three sectors to the input requirements from other sectors and final other demand may now be described by the matrix equation:

$$\begin{pmatrix} \frac{150}{600} & \frac{225}{900} & \frac{125}{300} \\ \frac{210}{600} & \frac{250}{900} & \frac{140}{300} \\ \frac{170}{600} & \frac{0}{900} & \frac{30}{300} \end{pmatrix} \begin{pmatrix} 600 \\ 900 \\ 300 \end{pmatrix} + \begin{pmatrix} 100 \\ 300 \\ 100 \end{pmatrix} = \begin{pmatrix} 600 \\ 900 \\ 300 \end{pmatrix}$$

This equation may be stated in general as:

$$AX + d = X$$

where X is the column of total outputs, d is the column of final (other) demands from outside the three sectors. The matrix A is called the matrix of technical coefficients: each column of A

gives the fraction of inputs to that sector which comes from each of the three sectors. This equation may be used to solve for the total output (X) required from each sector if the final demands (d) are changed:

$d = X - AX = (I - A)X$ ☺ **remember:** $IX = X$

$(I - A)^{-1}d = (I - A)^{-1}(I - A)X$ premultiply both sides by $(I - A)^{-1}$

$(I - A)^{-1}d = X$ since $(I - A)^{-1}(I - A) = I$

For the input/output model $AX + d = X$, the total output required from each sector when final demands d are changed is given by the equation:

$$X = (I - A)^{-1}d \qquad \textbf{(9.37)}$$

WORKED EXAMPLE 9.20

INPUT/OUTPUT ANALYSIS

Given the input/output table for the three-sector economy:

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300

If the final demands from each sector are changed to 500 from agriculture, 550 from industry, 300 from financial services, calculate the total output from each sector.

Solution

Step 1. Use the underlying assumption total input = total output to complete the input/output table.

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150	225	125	100	600
	Industry →	210	250	140	300	900
	Services →	170	0	30	100	300
	Other inputs	70	425	5		
	Total input	600	900	300		

Step 2. Calculate the matrix of technical coefficients, A, by dividing each column of inputs by total input.

		Input to				
		Agric.	**Industry**	**Services**	**Other demand**	**Total output**
Output from	Agric. →	150/600	225/900	125/300	100	600
	Industry →	210/600	250/900	140/300	300	900
	Services →	170/600	0/900	30/300	100	300
	Total input	600/600	900/900	300/300		

Step 3. Get the inverse of the matrix $(I - A)$, since this inverse matrix is required in the equation: $X = (I - A)^{-1}d$.

$$(I - A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{150}{600} & \frac{225}{900} & \frac{125}{300} \\ \frac{210}{600} & \frac{250}{900} & \frac{140}{300} \\ \frac{170}{600} & \frac{0}{900} & \frac{30}{300} \end{pmatrix} = \begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

To calculate the inverse of $(I - A)$, (i) use the elimination method, (ii) use the cofactor method. Set out the table of cofactors:

Elements of $(I - A)$	Cofactor = minor by (± 1)
0.75	$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix} \rightarrow \begin{vmatrix} 0.72 & -0.47 \\ 0.00 & 0.90 \end{vmatrix} (1) = 0.648$
−0.25	$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix} \rightarrow \begin{vmatrix} -0.35 & -0.47 \\ -0.28 & 0.90 \end{vmatrix} (-1) = 0.4466$
−0.42	$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix} \rightarrow \begin{vmatrix} -0.35 & 0.72 \\ -0.28 & 0.00 \end{vmatrix} (1) = 0.2016$

-0.35

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$|I - A| = 0.75(0.648) + (-0.25)(0.4466) + (-0.42)(0.2016) = 0.289678$$

$$\rightarrow \begin{vmatrix} -0.25 & -0.42 \\ 0.00 & 0.90 \end{vmatrix}(-1) = 0.225$$

0.72

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} 0.75 & -0.42 \\ -0.28 & 0.90 \end{vmatrix}(1) = 0.5574$$

-0.47

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} 0.75 & -0.25 \\ -0.28 & 0.00 \end{vmatrix}(-1) = 0.070$$

-0.28

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$= \begin{vmatrix} -0.25 & -0.42 \\ 0.72 & -0.47 \end{vmatrix}(1) = 0.4199$$

0.00

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ -0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$= \begin{vmatrix} 0.75 & -0.42 \\ -0.35 & -0.47 \end{vmatrix}(-1) = 0.4995$$

0.90

$$\begin{pmatrix} 0.75 & -0.25 & -0.42 \\ 0.35 & 0.72 & -0.47 \\ -0.28 & 0.00 & 0.90 \end{pmatrix}$$

$$= \begin{vmatrix} 0.75 & -0.25 \\ -0.35 & 0.72 \end{vmatrix}(1) = 0.4525$$

The inverse of $(I - A) = C^{T}/|I - A|$

$$= \frac{1}{0.289678} \begin{pmatrix} 0.648 & 0.4466 & 0.2016 \\ 0.225 & 0.5574 & 0.070 \\ 0.4199 & 0.4995 & 0.4525 \end{pmatrix}^{T}$$

$$= \frac{1}{0.289678} \begin{pmatrix} 0.648 & 0.225 & 0.4199 \\ 0.4466 & 0.5574 & 0.4995 \\ 0.2016 & 0.070 & 0.4525 \end{pmatrix}$$

Step 3. Finally, state the column of new external demands, d, and solve for X, by equation (9.34):

$$X = (I - A)^{-1}d$$

$$\begin{pmatrix} T_{\text{agi.}} \\ T_{\text{ind.}} \\ T_{\text{serv.}} \end{pmatrix} = \frac{1}{0.289678} \begin{pmatrix} 0.648 & 0.225 & 0.4199 \\ 0.4466 & 0.5574 & 0.4995 \\ 0.2016 & 0.070 & 0.4525 \end{pmatrix} \begin{pmatrix} 500 \\ 550 \\ 300 \end{pmatrix} = \begin{pmatrix} 1980.5 \\ 2346.5 \\ 949.5 \end{pmatrix}$$

Progress Exercises 9.6 The Inverse Matrix and Input/Output

In the following questions, determine the inverse matrices using (i) the elimination method and (ii) the cofactor method.

1. Determine the inverse of the following matrices:

(a) $\begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 4 \\ 20 & 10 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 4 \\ 12 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 4 \\ -2 & -8 \end{pmatrix}$

In questions 2, 3, 4 and 5,

(a) Determine the inverse of the matrix.
(b) Use the inverse matrix method to solve the given system of equations.

2. (a) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 2 \\ 6 & 3 & 6 \end{pmatrix}$ (b) $\begin{aligned} x + y &= 12 \\ 2x + 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$

3. (a) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & -5 & 2 \\ 6 & 3 & 6 \end{pmatrix}$ (b) $\begin{aligned} x + y &= 12 \\ 2x - 5y + 2z &= 20 \\ 6x + 3y + 6z &= 0 \end{aligned}$

4. (a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$ (b) $\begin{aligned} x + y &= 2 \\ x - y + 2z &= 20 \\ 2x + 3y + z &= 0 \end{aligned}$

5. (a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -5 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ (b) $\begin{aligned} x + y + z &= 12 \\ 2x - 5y + 2z &= 20 \\ x + z &= 0 \end{aligned}$

6. Given the matrix of technical coefficients and the matrix of final demand (other demand),

(i) $A = \begin{pmatrix} 0.5 & 0.4 \\ 0.25 & 0.4 \end{pmatrix}$, $B = \begin{pmatrix} 40 \\ 100 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 0.5 & 0.25 \\ 0.2 & 0.5 \end{pmatrix}$, $B = \begin{pmatrix} 120 \\ 400 \end{pmatrix}$

(a) Determine $(I - A)^{-1}$.
(b) Calculate the total output required from each sector.

7. Given the matrix of technical coefficients, A, and final demand (other demand) B for industries 1, 2, and 3,

$$A = \begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.6 \end{pmatrix}, \quad B = \begin{pmatrix} 120 \\ 200 \\ 700 \end{pmatrix}$$

(a) Determine $(I - A)^{-1}$.
(b) Calculate the total output required from each sector.

8. Repeat question 7 for the following matrices:

$$A = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.2 & 0.25 & 0.1 \\ 0.4 & 0.2 & 0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 1020 \\ 1200 \\ 700 \end{pmatrix}$$

9. Given the inter-industrial transaction table for two industries,

	Input to X	Input to Y	Final demand
Output from X	600	400	200
Output from Y	600	200	0

(a) Write down the matrix of technical coefficients.
(b) Calculate the total output required from each industry if the final demands from X and Y change to 500 and 1000 units respectively.

10. The input/output table for an organic farm is given below:

		Input to				
		Hortic.	**Dairy**	**Poultry**	**Other demand**	**Total output**
Output from	Hortic →	50	40	200	210	500
	Dairy →	100	160	0	140	400
	Poultry →	200	0	80	520	800

(a) Write down the matrix of technical coefficients.
(b) Calculate the total output required from each section when other demands change to 200, 800, 1000 from horticulture, dairy and poultry sections respectively.

11. The input/output table for a three-sector industry is given below:

		Input to A	B	C	Other demand	Total output
Output from	A →	200	100	200	500	1000
	B →	100	0	500	200	800
	C →	100	80	0	620	800

(a) Write down the matrix of technical coefficients.
(b) Calculate the total output required from each industry when other demands change to 400, 400, 1000 from sectors A, B and C respectively.

12. The input/output table for three inter-dependent industries is given below:

		Input to Dairy	Beef	Leather	Other demand	Total output
Output from	Dairy	400	200	0	400	1000
	Beef	100	400	200	200	1000
	Leather	200	200	100	600	1000

(a) Write down the matrix of technical coefficients.
(b) Calculate the total output required from each industry when other demands change to 750, 200, 1500 from dairy, beef and leather industries respectively.

9.6 Excel for Linear Algebra

The elimination methods involve the use of elementary row operations. Examples of elementary row operations include (i) multiplying rows by non-zero constants and (ii) adding multiples of rows to other rows. Excel is ideal for these calculations. The following worked example should demonstrate the usefulness of Excel in carrying out the tedious, error-prone calculations required for elimination methods. Excel is also useful for the evaluation of determinants, hence in solving equations by Cramer's rule.

WORKED EXAMPLE 9.21

USE EXCEL TO SOLVE SYSTEMS OF LINEAR EQUATIONS

(a) Find the inverse of

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 5 & -3 \\ 4 & -1 & 3 \end{pmatrix}$$

by the elimination method, hence solve the equations

$$2x + y + z = 12$$
$$6x + 5y - 3z = 6$$
$$4x - y + 3z = 5$$

(b) Solve the equations in (a) by Gauss–Jordan elimination.

Solution

Set up the augmented matrix, consisting of the matrix A and the unit matrix of the same dimension. Add multiples of rows to other rows to reduce A to upper triangular form. Enter the Excel formula for each row operation in column 1, then copy the formula across the remaining five columns. See Chapter 1, and remember that formulae start with =.

For example, to divide row 1 by 2 , place the cursor in cell B8 and type = B4/2. To copy this formula across, click on the corner of cell B8 until a black cross appears, then drag this across the remaining five columns. This updated row is labelled row 1^1. Similarly, to add row $1^1 \times (-6)$ to row 2 in order to generate the required 0, place the cursor in cell B9 and type in the formula = B5 + B8 * (−6). Copy the formula across the row. In this way carry out all the row operations as indicated.

1	A	B	C	D	E	F	G	
2	Augmented matrix							
3		col 1	col 2	col 3	col 4	col 5	col 6	
4	row 1 →	2	1	1	1	0	0	
5	row 2 →	6	5	-3	0	1	0	
6	row 3 →	4	-1	3	0	0	1	
7								Operation **Formula**
8	row 1^1 →	1	0.5	0.5	0.5	0	0	row 1 divided by 2 = **B4/2**
9	row 2^1 →	0	2	-6	-3	1	0	row 2 + row $1^1 \times (-6)$ = **B5 + B8 * (−6)**
10	row 3^1 →	0	-3	1	-2	0	1	row 3 + row $1^1 \times (-4)$ = **B6 + B8 * (−4)**
11								
12	row 1^1 →	1	0.5	0.5	0.5	0	0	
13	row 2^2 →	0	1	-3	-1.5	0.5	0	row 2^1 divided by 2 = **B12/2**
14	row 3^2 →	0	0	-8	-6.5	1.5	1	row 3^1 + row $2^2 \times (3)$ = **B10 + B13 * (3)**

A is reduced to upper triangular form: row 1^1, row 2^2 and row 3^2. Continue with the row operations to further reduce A to the unit matrix.

15	row $3^3 \rightarrow$	0	0	1	0.81	-0.2	-0.1	row 3^2 divided by -8 = **B14**/(**−8**)
16	row $1^2 \rightarrow$	1	0.5	0	0.09	0.09	0.06	row 1^1 + row $3^3 \times (-0.5)$ = **B12** + **B15** * (**−0.5**)
17	row $2^3 \rightarrow$	0	1	0	0.94	-0.1	-0.4	row 2^2 + row $3^3 \times (3)$ = **B13** + **B15** * (**3**)
18	row $1^3 \rightarrow$	1	0	0	-0.4	0.13	0.25	row 1^2 + row $2^3 \times (-0.5)$ = **B16** + **B17** * (**−0.5**)

The reduction is now complete. Write out the most recently updated rows in order.

	col 1	col 2	col 3	col 4	col 5	col 6
row $1^2 \rightarrow$	1	0	0	-0.375	0.0125	0.025
row $2^3 \rightarrow$	0	1	0	0.9375	-0.0625	-0.375
row $3^3 \rightarrow$	0	0	1	0.8125	-0.8125	-0.125

Read off the inverse matrix: columns 4, 5 and 6. The inverse of A is

-0.375	0.125	0.25
0.9375	-0.0625	-0.375
0.8125	-0.1875	-0.125

Finally, to multiply A^{-1} by the column of constants from the RHS of the equations, arrange these as shown.

	B	C	D	E	
19	Inverse of A			RHS	
20	-0.375	0.125	0.25	12	
21	0.9375	-0.0625	-0.375	6	
22	0.8125	-0.1875	-0.125	5	
23					
24	x = -2.5	= **B20** * **E20** + **C20** * **E21** + **D20** * **E22**			
25	y = 9	= **B21** * **E20** + **C21** * **E21** + **D21** * **E22**			Formulae used to calculate x, y and z
26	z = 8	= **B22** * **E20** + **C22** * **E21** + **D22** * **E22**			

(b) To solve by Gauss–Jordan elimination, go back to the augmented matrix in (a) and replace col 5 (first column of the unit matrix) by the column of constants from the RHS of each equation. Delete col 5 and col 6 (see below). To carry out the Gauss–Jordan elimination, copy across the formulae that were entered earlier to find the inverse of A. Read off the solution.

	Augmented matrix			
row 1	2	1	1	12
row 2	6	5	-3	6
row 3	4	-1	3	5
row 1^1	1	0.5	0.5	6
row 2^1	0	2	-6	-30
row 3^1	0	-3	1	-19
row 2^2	0	1	-3	-15
row 3^2	0	0	-8	-64

row 1 divided by 2
row 2 + row $1^1 \times (-6)$
row 3 + row $1^1 \times (-4)$
row 2^1 divided by 2
row 3^1 + row $2^2 \times (3)$

At this stage it is possible to read off the upper triangular matrix: row 1^1, row 2^2 and row 3^2. Solve by back substitution. For Gauss–Jordan elimination continue the row operations.

row 3^3	0	0	1	8
row 1^2	1	0.5	0	2
row 2^3	0	1	0	9
row 1^3	1	0	0	-2.5

row 3^2 divided by -8
row 1^1 + row $3^3 \times (-0.5)$
row 2^2 + row $3^3 \times (3)$
row 1^2 + row $2^3 \times (-0.5)$

Read off the solution from the reduced augmented matrix: $x = -2.5$, $y = 9$, $z = 8$.

	col 1	col 2	col 3	col 4
row $1^2 \rightarrow$	1	0	0	-2.5
row $2^3 \rightarrow$	0	1	0	9
row $3^3 \rightarrow$	0	0	1	8

9.7 Summary

❒ **Linear programming**

Find the optimum value of a linear function (objective function) subject to two or more linear constraints.

Method 1: Graph the inequality constraints and one level of the objective function. Determine graphically the largest value (or smallest value for minimisation) of the objective function

which contains a corner point, X, in the feasible region. The coordinates of the point X are the values of the variables for which the objective function is optimised.

Method 2: Calculate the corner points of the feasible region. Evaluate the objective function at each corner point, to obtain the point at which it is optimised.

❒ Matrix algebra

1. A matrix is a rectangular array of numbers (elements).
2. A matrix of order $m \times n$ contains m rows and n columns.
3. Matrices added by adding corresponding elements and subtracted by subtracting corresponding elements, therefore the matrices must be of the same order.
4. A matrix is transposed by writing each row as the corresponding column.
5. Scalar multiply matrices by multiplying each element in the matrix by the scalar (the constant).
6. The unit matrix is a square matrix with zeros everywhere except on the diagonal; the diagonal elements are usually 1.
7. The product AB of matrices A and B is possible only when the number of columns in A is equal to the number of rows in B. In general $AB \neq BA$.
8. Gauss–Jordan elimination is a method for solving a system of equations.

❒ Determinants

1. A determinant is a square array of numbers (elements).
2. The value of a 2×2 determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3. The value of a 3×3 determinant is given by

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

4. Cramer's rule: to solve two simultaneous equations

$$a_1x + b_1y = d_1$$
$$a_2x + b_2y = d_2$$

use determinants

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta_x}{\Delta} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta_y}{\Delta}$$

Cramer's rule can be extended to the solution of three linear equations in three unknowns (x, y, z):

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

❒ The inverse of a square matrix

1. The minor of an element is the determinant of what remains when the row and column containing that element is crossed out.
2. The cofactor of an element is the value of the minor multiplied by ± 1. The ± 1 is given by the determinant of signs.
3. Given a matrix A, the inverse of A is defined as

$$A^{-1} = \frac{C^{\mathrm{T}}}{|A|}$$

where C^T is the matrix in which every element is replaced by its cofactor.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{1.1} & C_{1.2} & C_{1.3} \\ C_{2.1} & C_{2.2} & C_{2.3} \\ C_{3.1} & C_{3.2} & C_{3.3} \end{pmatrix}^{\mathrm{T}}$$

To find the inverse by Gauss–Jordan elimination, set up the augmented matrix as $(A|I)$, where A is the square matrix to be inverted and I is the unit matrix of the same dimension. Use Gauss–Jordan elimination to reduce A to the unit matrix, hence $(A|I) \rightarrow (I|A^{-1})$. Read off the inverse of A.

❒ Applications of inverse matrices

1. Simplifying calculations and other operations on large arrays of data.
2. Solving simultaneous equations.

❒ Input/output analysis

Input/output analysis is based on the condition

$$\text{total input} = \text{total output}$$

For the input/output model $AX + d = X$, the total output required from each sector when final demands d are changed to d_{new} is given by the equation

$$X = (I - A)^{-1} d_{\text{new}}$$

where

A = the matrix of technical coefficients (the elements in each column are the proportion of that sector's input derived from each other sector)
d = the column matrix of final demands from each sector
X = the column matrix of total output from each sector

Test Exercises 9

1. Graph the inequality constraints and shade in the feasible region:

(a) $10x + 2.5y \leq 50$: $2x + 12y \leq 24$: $x \geq 0$: $y \geq 0$

(b) $4x + 5y \geq 50$: $2x + 12y \geq 24$: $x \geq 0$: $y \geq 0$

2. A small iron works manufactures two types of gate. The requirements for each stage of production, along with the limitations on the available labour-hours are given in the following table:

	Stages of production				
Gate type	**Welding**	**Finishing**	**Sales**	**Selling price**	**Unit profit**
Security	4.5	1.0	1.0	£7200	£180
Ornamental	2.0	2.0	1.0	£665	£315
Max. hours	900	400	250		

(a) Write down the equations for (i) profit, (ii) total revenue.
(b) Write and graph the inequality constraints. Hence determine graphically the number of each type of gate which should be produced and sold to maximise
(i) profit and (ii) total revenue.
(c) Confirm the answers in (b) algebraically.

3. Given the matrices

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}: \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 4 \\ -2 & 2 \end{pmatrix}: \quad D = \begin{pmatrix} -5 & -5 \\ 0 & 10 \end{pmatrix}$$

Calculate the following, if possible (if not possible, give reasons):

(a) $A + D$ (b) CD (c) AB (d) $A + C$ (e) AC^{T}

4. State Cramer's rule. Hence
(a) Solve the following simultaneous equations:
(i) $5x - 8y = 55$: $10x + 2y = 20$ (ii) $3Q + 12P + 60 = 0$: $10Q = 4P + 200$
(b) The equilibrium condition in
(i) The goods market is given as $Y = C + I$, where $C = 330 + 0.2Y$: $I = 150 - 0.4r$
(ii) The money market is $M_{\mathrm{d}} = M_{\mathrm{s}}$, where $M_{\mathrm{d}} = 100 + 0.1Y - 0.2r$: $M_{\mathrm{s}} = 156$
Write each equilibrium condition in the form $aY + br = c$. Hence use Cramer's rule to solve for the equilibrium levels of Y and r.

5. What is the basic assumption underlying the input–output model?
The input–output table for a three-sector economy is given in the following table:

		Sector A	**Input to Sector B**	**Sector C**	**Final demand**	**Total output**
Output	Sector A	15	5	0	30	50
from	Sector B	5	15	10	10	40
	Sector C	0	0	10	70	80

(a) Write out the matrix of technical coefficients.
(b) Calculate the total output required from each sector if the final demand changes to 10 from A, 20 from B and 100 from C.

6. The input–output table for a two-sector economy is given as follows:

		Input to sector A	**Input to sector B**	**Final demand**
Output from	Sector A	200	50	50
Output from	Sector B	100	400	100

(a) Write out the matrix of technical coefficients.
(b) Calculate the total output required from each sector when final demand increases to 200 from sector A and 250 from sector B.
(c) Confirm your answer by writing out the full input–output table.