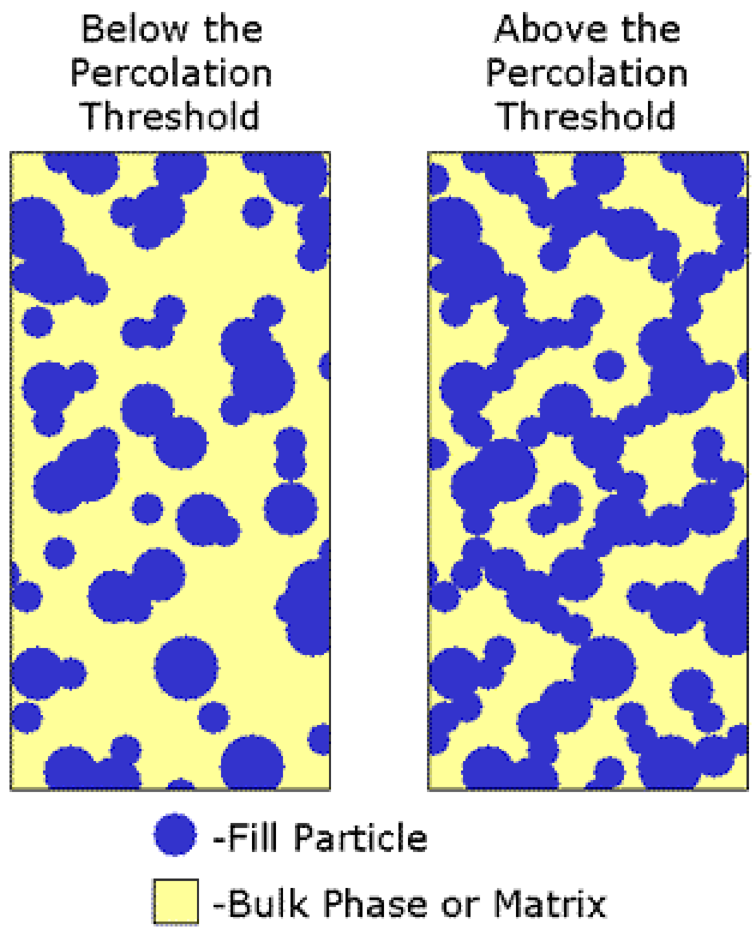
Topology Control (I)

4/21/06

On a rainy day

- Observe the raindrops falling on the pavement. Initially the wet regions are isolated and we can find a dry path. Then after some point, the wet regions are connected and we can find a wet path.
- There is a critical density where sudden change happens.



Phase transition

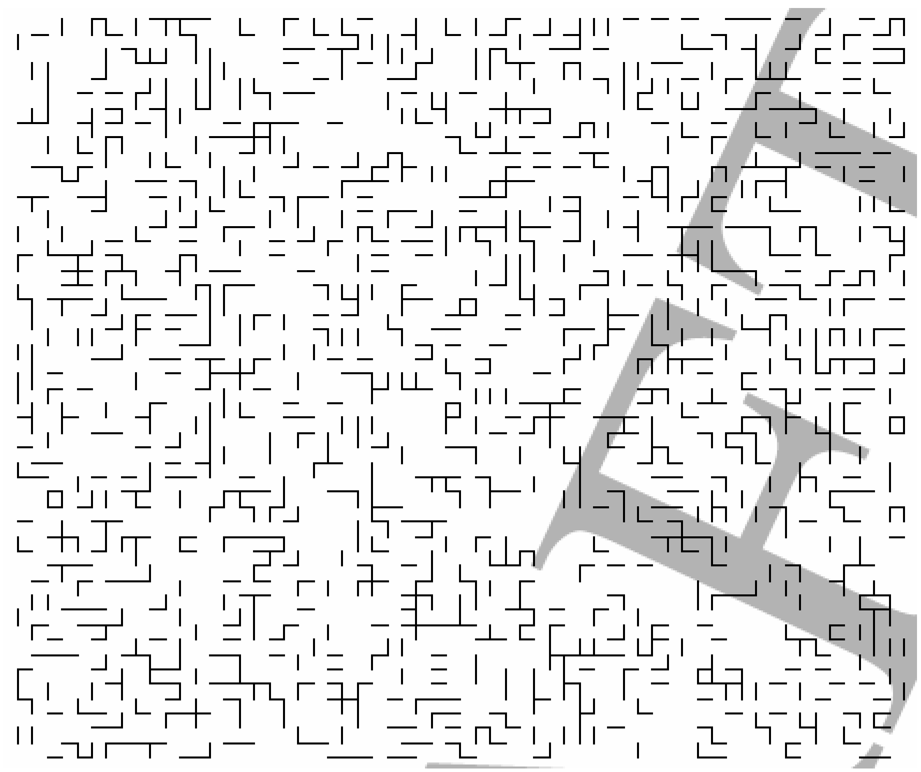
- In physics, a **phase transition** is the transformation of a thermodynamic system from one **phase** to another. The distinguishing characteristic of a **phase transition** is an abrupt sudden change in one or more physical properties, in particular the heat capacity, with a small change in a thermodynamic variable such as the temperature.

- Solid, liquid, and gaseous phases.
- Different magnetic properties.
- Superconductivity of medals.

- This generally stems from the interactions of an extremely large number of particles in a system, and does not appear in systems that are too small.

Bond Percolation

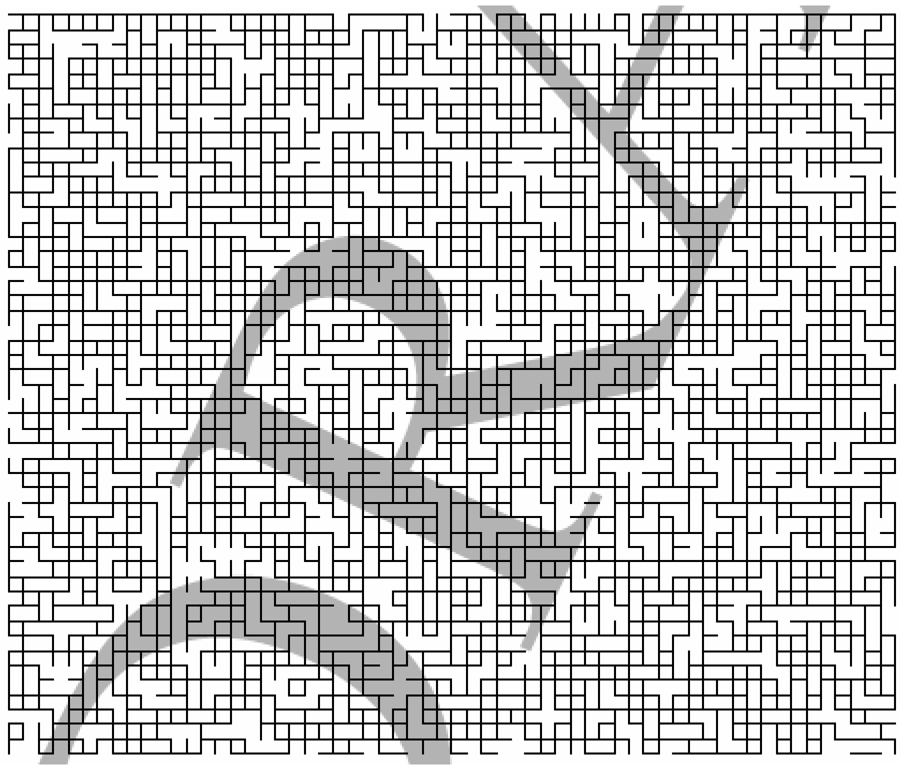
- An infinite grid Z^2, with each link to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.



p=0.25

Bond Percolation

- An infinite grid Z^2, with each link to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.

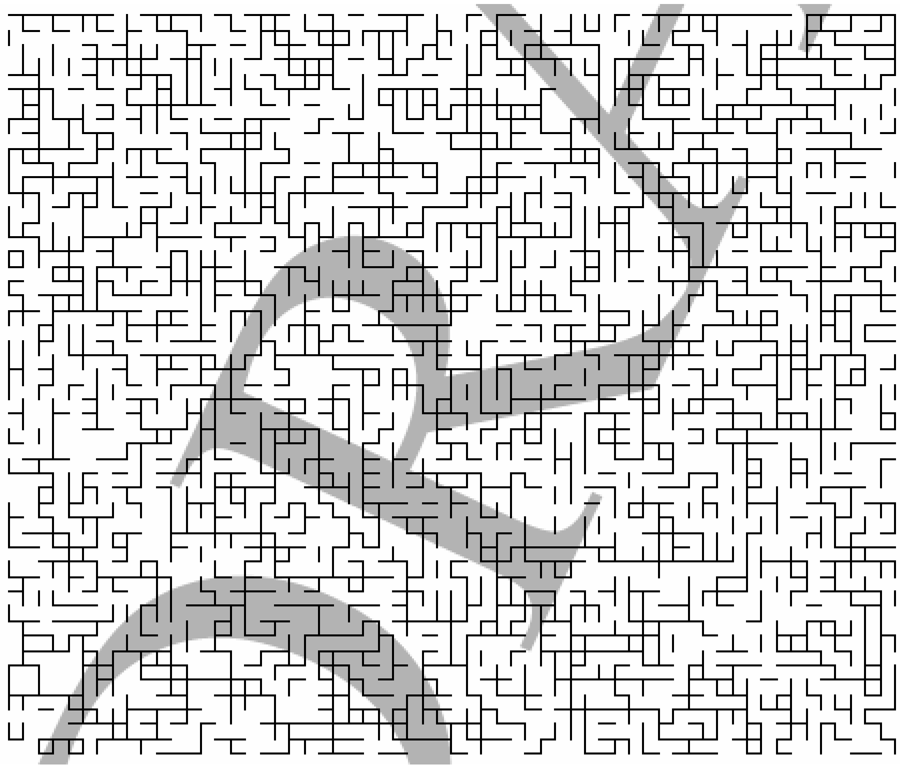


p=0.75

Bond Percolation

- An infinite grid Z^2, with each link to be “open” (appear) with probability p independently. Now we study the connectivity of this random graph.

No path from left to right

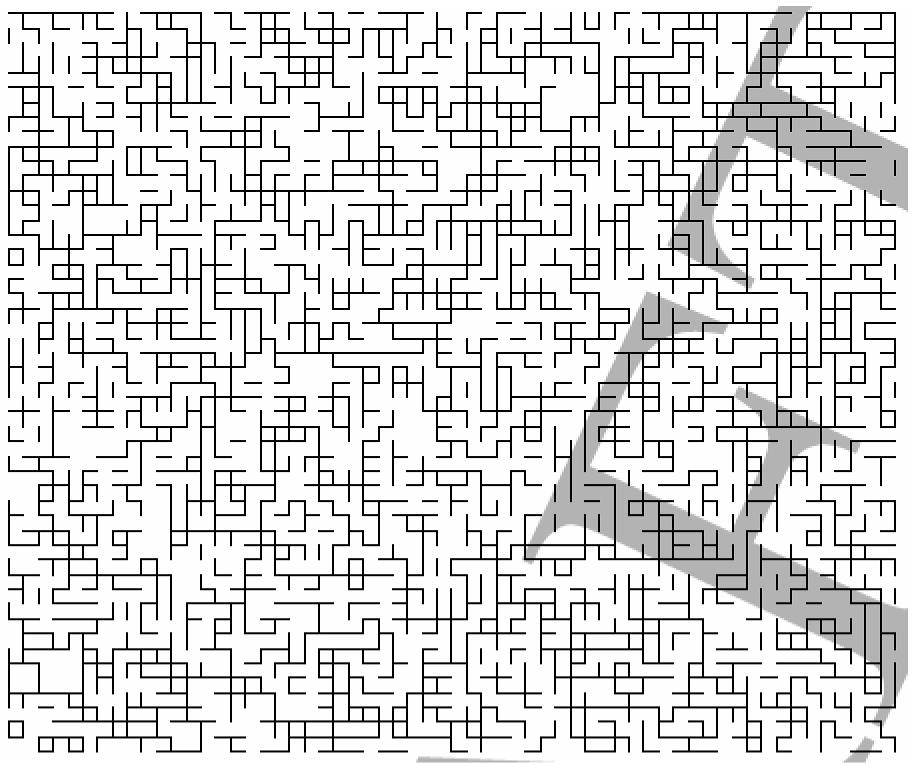


p=0.49

Bond Percolation

- An infinite grid Z^2, with each link to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.

There is a path from left to right!

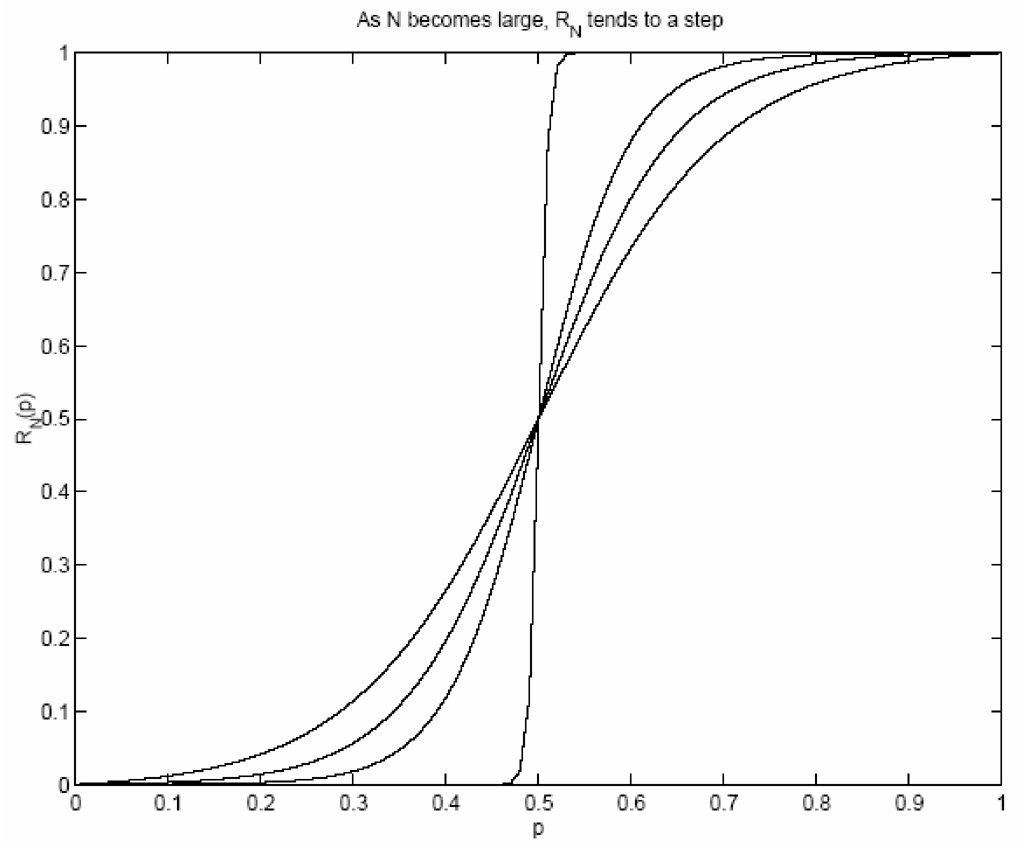


p=0.51

Bond Percolation

- There is a critical threshold p=0.5.

The probability that there is a "bridge" cluster that spans from left to right.



Bond Percolation

- There is a critical threshold p=0.5.
- When p>0.5, there is a unique infinite size cluster almost always.
- When p<0.5, there is no infinitely size cluster.
- When p=0.5, the critical value, there is no infinite cluster.

- Percolation theory studies the phase transition in random structures.

Main problems in percolation

- What is the critical threshold for the appearance of some property, e.g., an infinite cluster?

- What is the behavior below the threshold? We know all clusters are finite. How large are they? Distribution of the cluster size?

- What is the behavior above the threshold? We know there exists an infinite cluster? Is it unique? What is the asymptotic size with respect to p and n (the network size)?

- What is the behavior at the threshold? Is there an infinite cluster or not? What is the size of the clusters?

Examples of Percolation

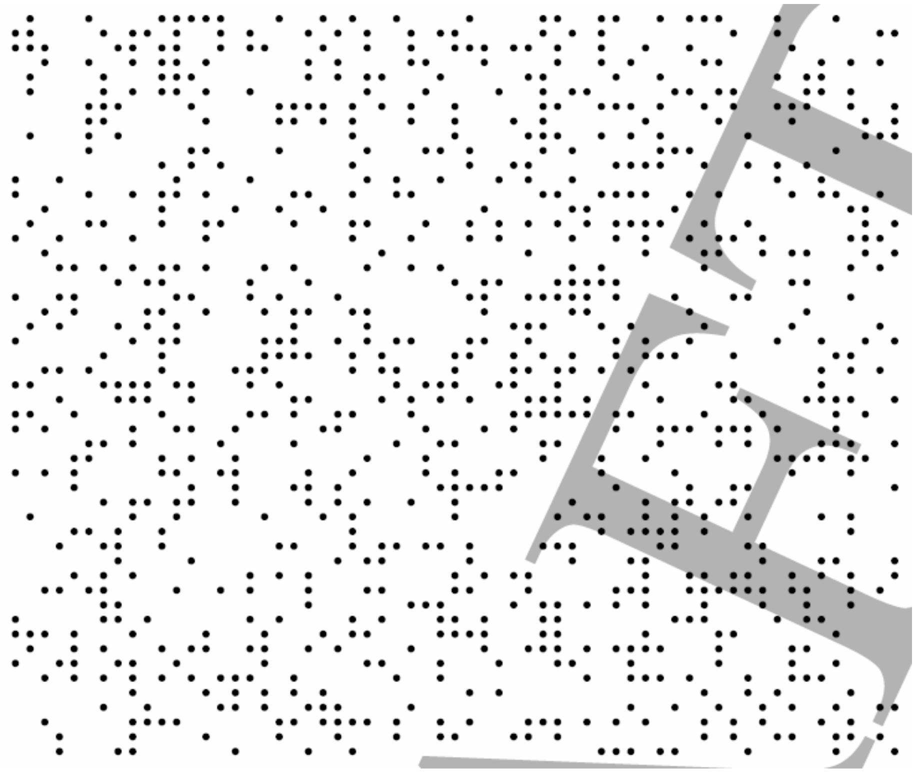
- **Spread of epidemics, virus infection on the Internet.**
  - Each “sick” node has probability p to infect a neighbor node.
  - Denote by p the contagious parameter. If p is above the percolation threshold, then the disease will spread world wide.
  - The real model is more complicated, taking into account the time variation, healing rate, etc.
- **Gossip-based routing, content distribution in P2P network, software upgrade.**
  - The graph is important in deciding the critical value.
  - An interesting result is about the “scale-free” graphs (also called power-law) that model the topology of the Internet or social network: in one of such models (random attachment with preferential rule), the percolation threshold vanishes.

More examples

- **Connectivity of unreliable networks.**
  - Each edge goes down randomly.
  - Is there a path between any two nodes, with high probability?
  - Resilience or fault tolerance of a network to random failures.
- **Random geometric graph, density of wireless nodes (or, critical communication range).**
  - Wireless nodes with Poisson distribution in the plane.
  - Nodes within distance r are connected by an edge.
  - There is a critical threshold on the density (or the communication range) such that the graph has an infinitely large connected component.

Site Percolation

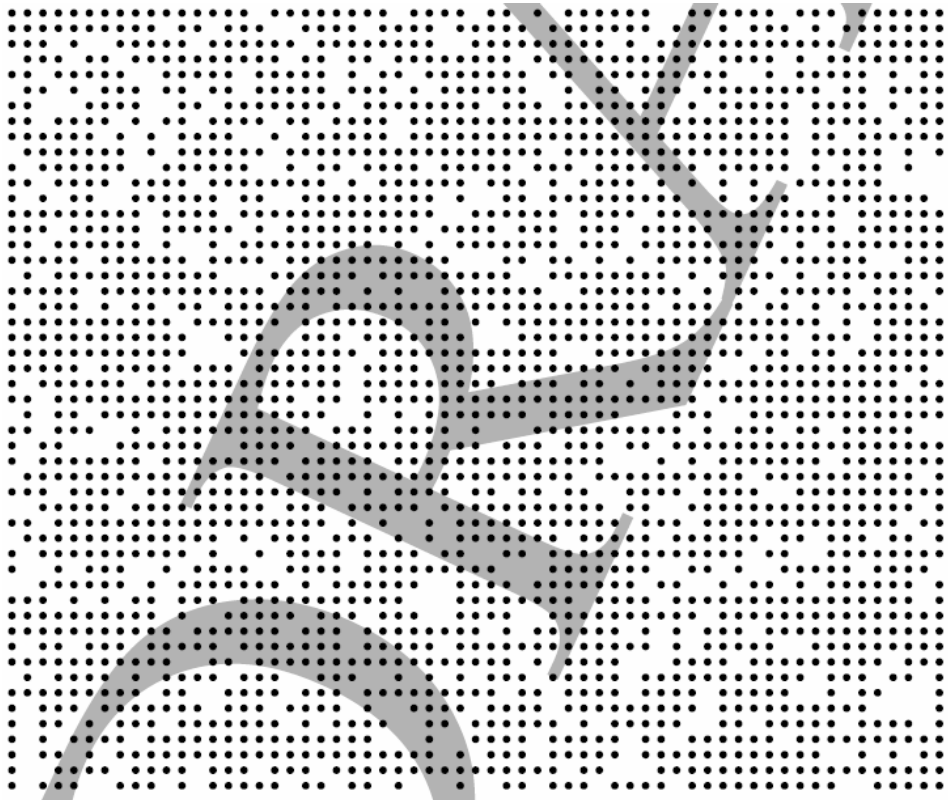
- An infinite grid Z^2, with each vertex to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.



p=0.3

Site Percolation

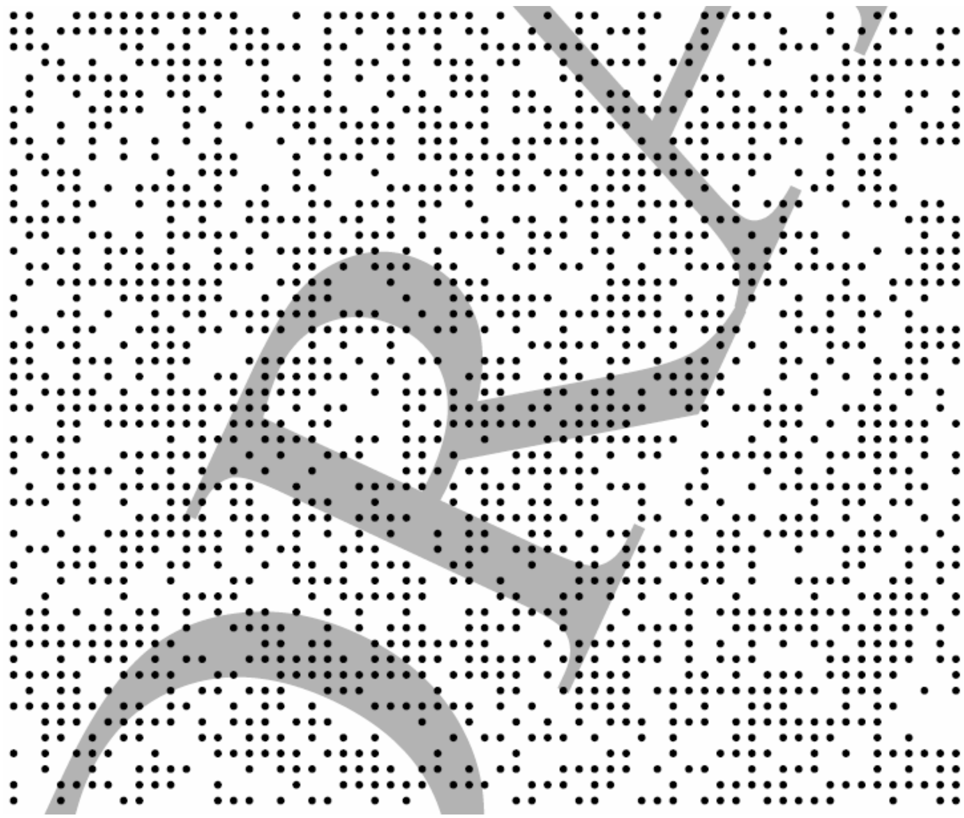
- An infinite grid Z^2, with each vertex to be "open" (appear) with probability p independently. Now we study the connectivity of this random graph.



p=0.80

Site Percolation

- Percolation threshold is still unknown. Simulation shows it's around 0.59. (note this is larger than bond percolation)



p=0.58

Site Percolation

- Site percolation is a generalization of bond percolation.
- Every bond percolation can be represented by a site percolation, but not the other way around.

- Percolation in an infinite connected graph G(V, E).
- Bond percolation: each edge appears with probability p.
- Site percolation: each vertex appears with probability p.

- Denote an arbitrary node as origin, study the cluster containing the origin.

- The percolation threshold of site percolation is always larger than bond percolation.

Continuum Percolation

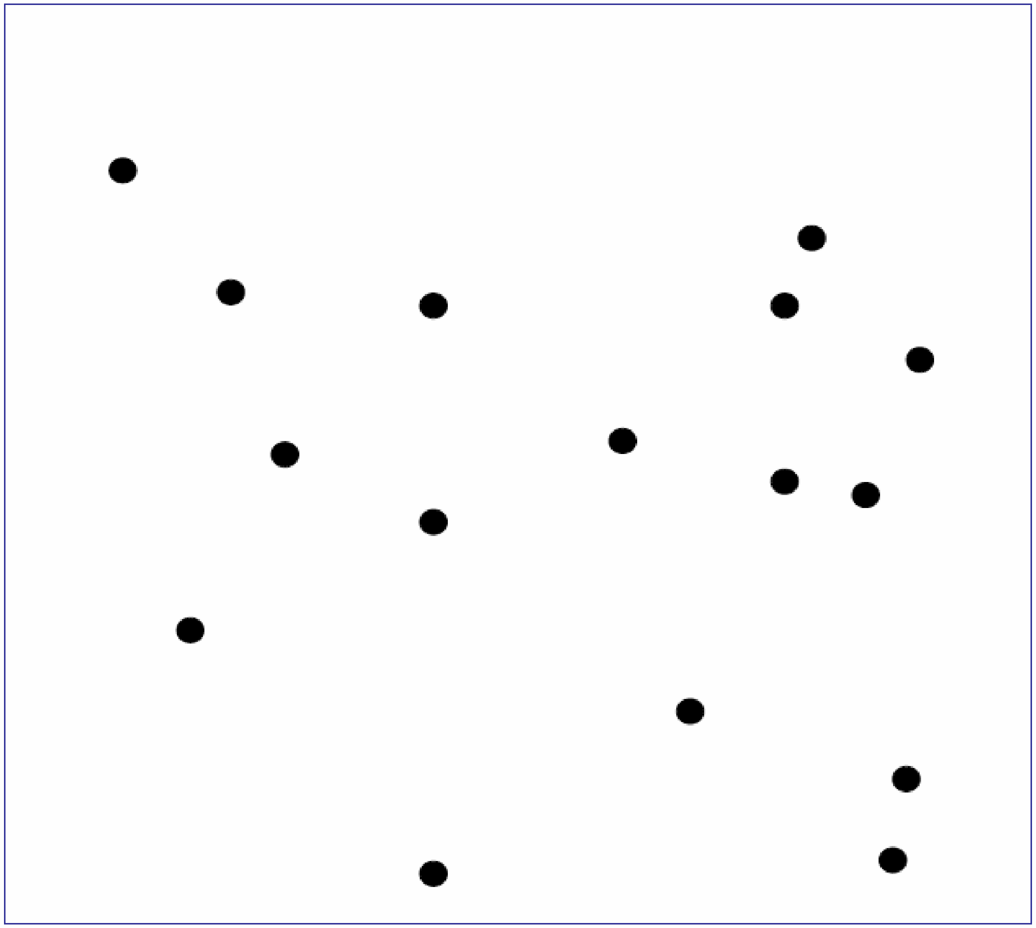
- **Random plane network**, by Gilbert, in J. SIAM 1961.

- Pick points from the plane by a Poisson process with density λ points per unit area.
- Join each pair of points if they are at distance less than r.

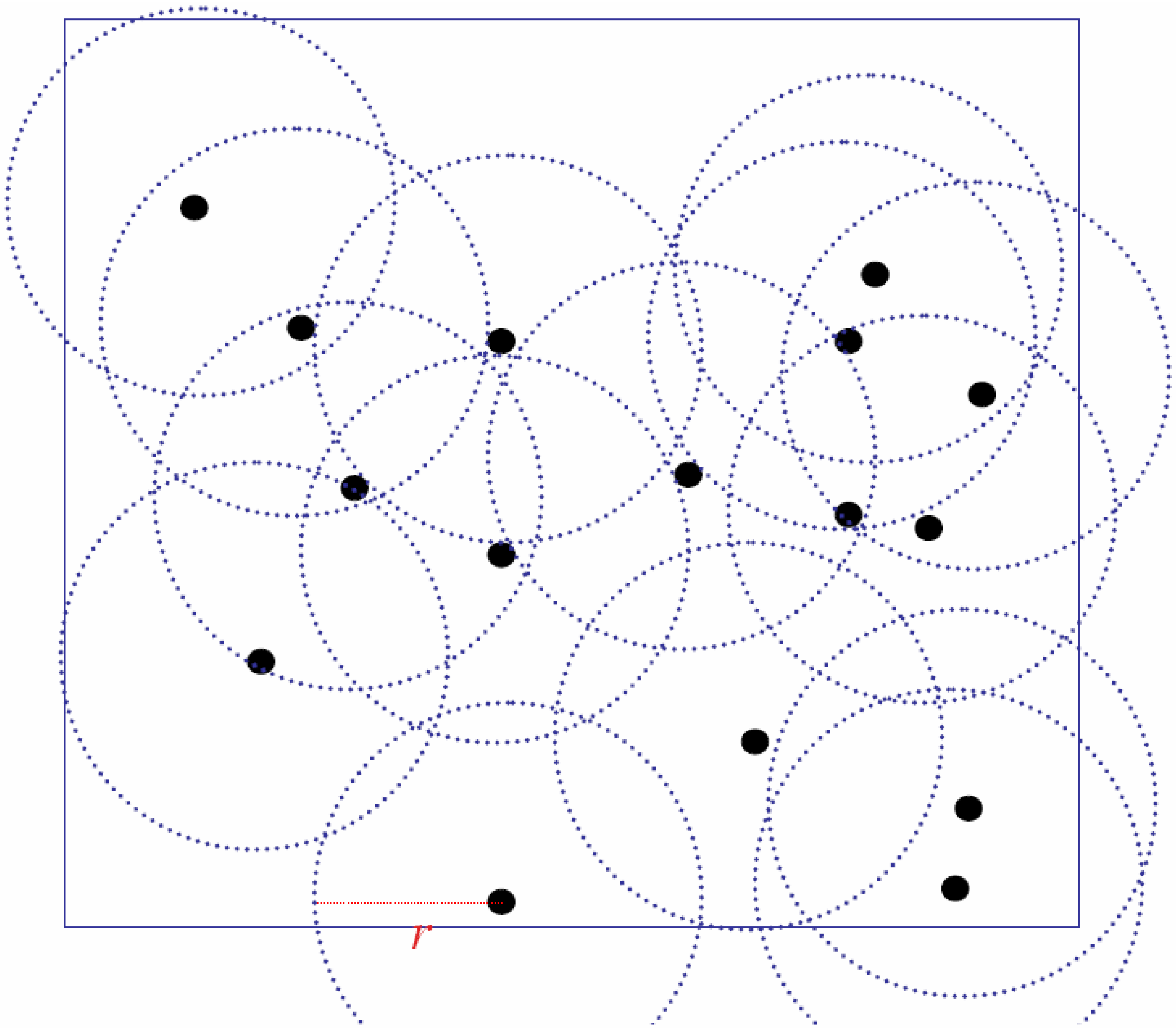
- Equivalently,
- In the unit square [0, 1] by [0, 1], throw n points uniformly randomly.
- Connect two nodes with distance less than r.

- This graph is denoted as G(n, r).

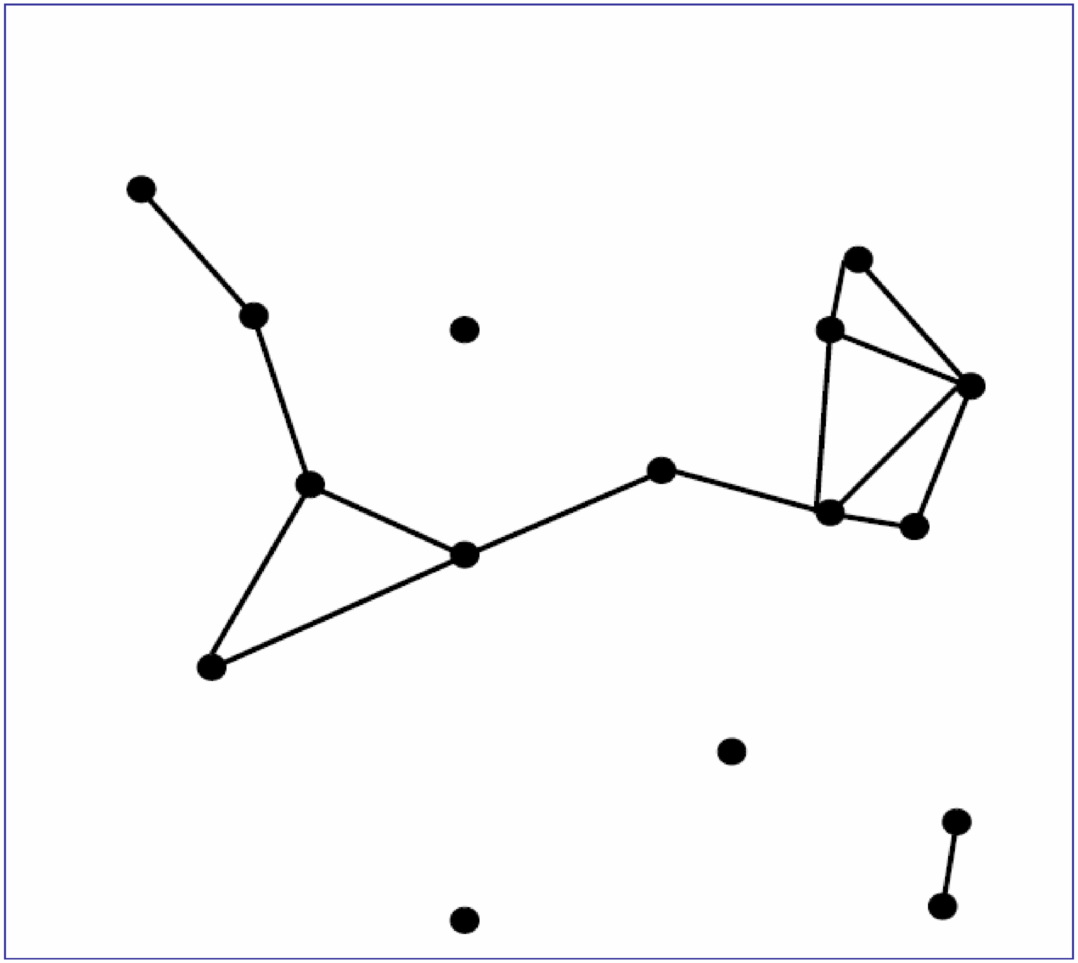
Random geometric graph



Random geometric graph



Random geometric graph



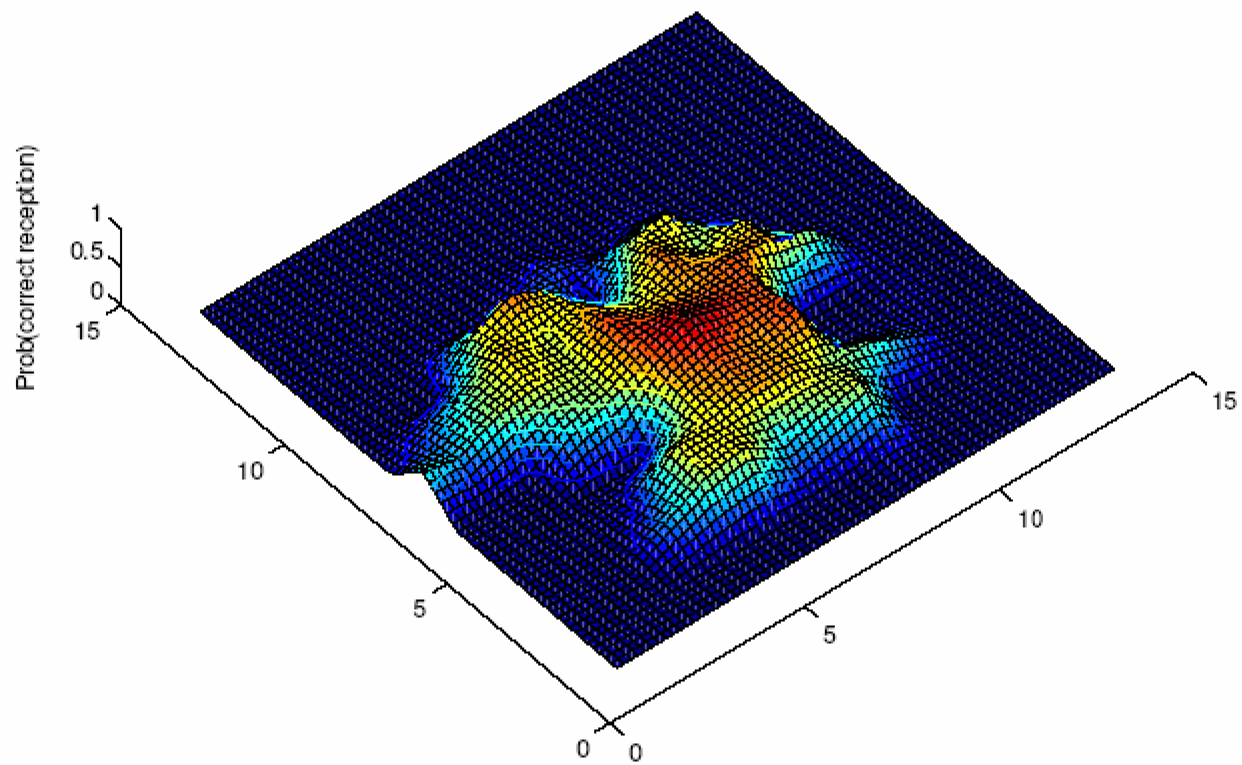
Random geometric graph

- Percolation behavior:
- Given G(n, r), and a desired property (e.g., connectivity), we want to find the smallest radius $r_Q(n)$ such that Q holds with high probability.

- Gupta and Kumar proved:
- Connectivity: if $\pi rn^2 = (\log n + c_n)/n$.
- As c_n goes to infinity, the graph is almost surely connected.
- As c_n goes to –infinity, the graph is almost surely disconnected.

Percolation in the real world?

- Communication range is not a perfect disk.



Percolation with noisy links

- Each pair of nodes is connected according to some (probabilistic) function of their (random) positions.
- A pair of points (i, j) is connected with probability $g(x_i - x_j)$, where g is a general function that depends only on the distance.

- In order to keep the average degree the same, fix the effective area $e(g) = \int_{x \in \mathbb{R}^2} g(x)dx$
- The average degree = λ e(g).

Percolation with noisy links

- Percolation threshold

$$0 < \lambda_c(g) = \inf\{\lambda : \exists \text{ infinite connected component a.s.}\} < \infty.$$

- Question: what is the relationship between the percolation threshold and the function g?

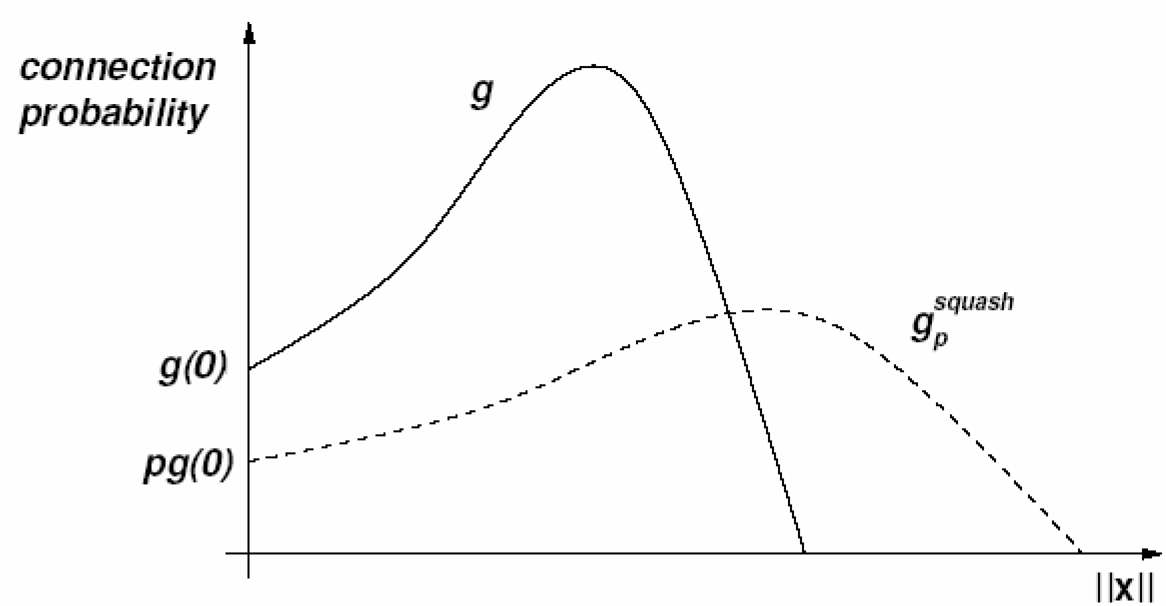
Percolation with noisy links

- Question: what is the relationship between the percolation threshold and the function g?
- Each node is connected to the same number of edges on average. So whom should the node be connected to, in order to have a small percolation threshold?
- Which distribution has the best graph connectivity?
- Should I use reliable short links? Or unreliable long links? Or something more complex, say an annulus?

Squashing

- Probabilities are reduced by a factor of p, but the function is spatially stretched to maintain the same effective area (e.g., the same average degree).

$$g_p^{squash}(x) = p \cdot g(\sqrt{p}x).$$



Squashing

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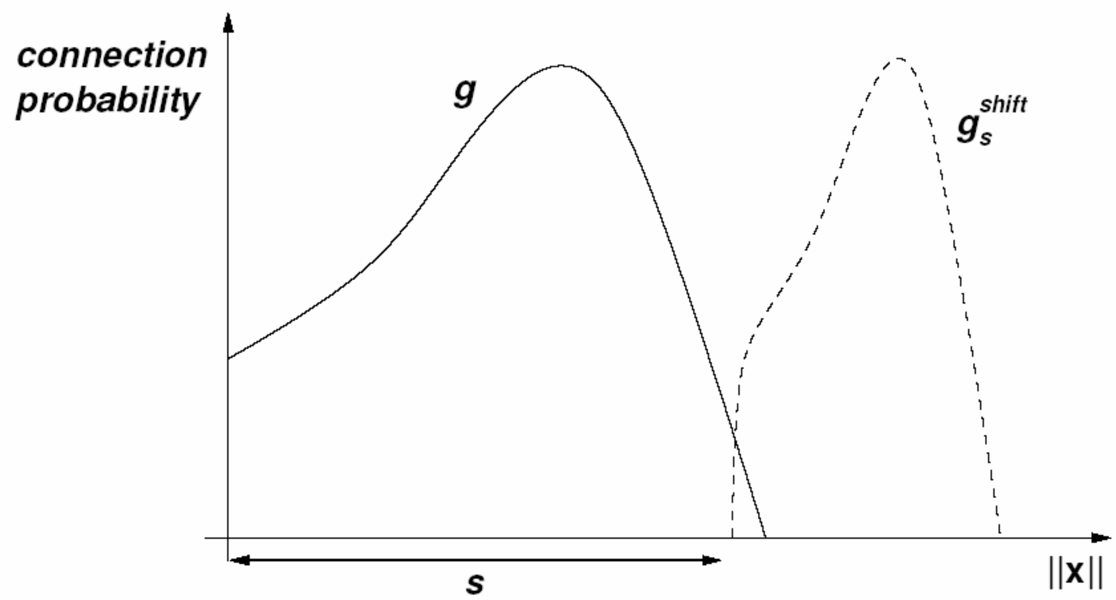
$$g_p^{squash}(x) = p \cdot g(\sqrt{p}x).$$

- Theorem: $\lambda_c(g) \geq \lambda_c(g_p^{squash}).$
- It's beneficial for the connectivity to use long unreliable links!
- If the effective area is spread out, then the threshold density goes to 1.
- Question: what makes the difference? The guess is the existence of long links.

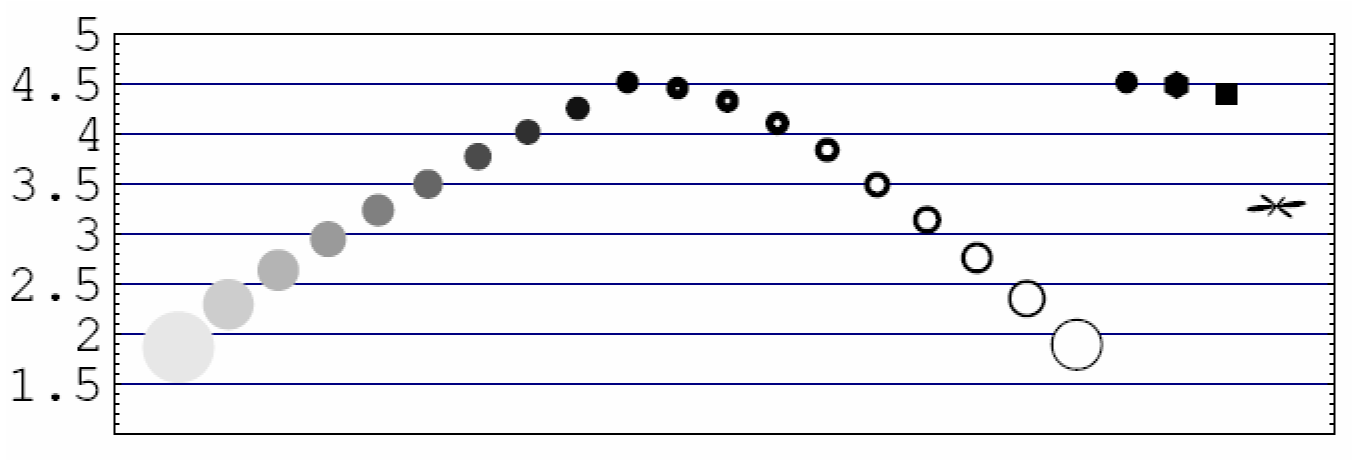
Shifting and squeezing

- Shift the function g outward by a distance s, but squeeze the function after that, so that it has the same effective area.
- Goal: use long links.



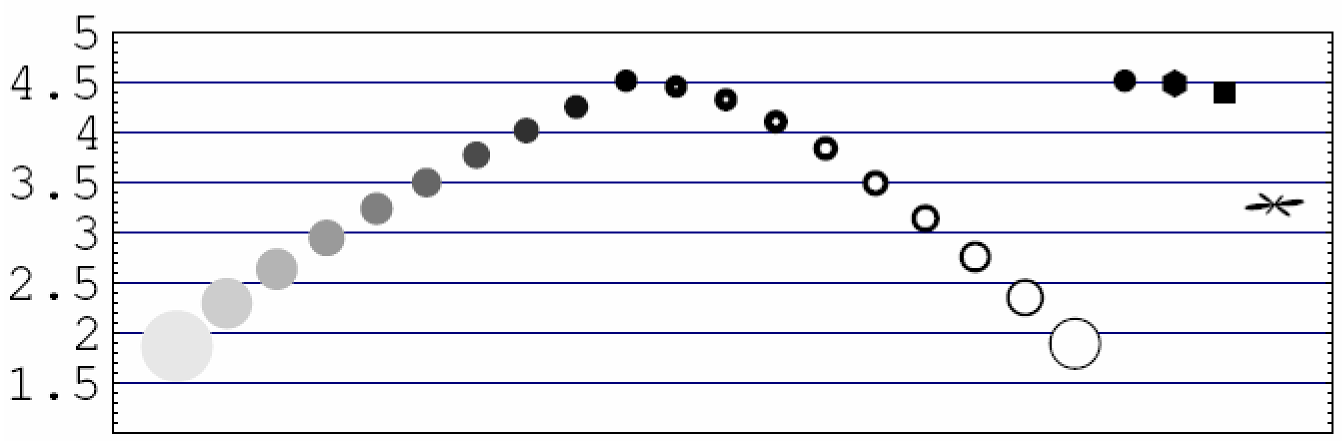
Shifting and squeezing

- Yes it helps percolation! The density threshold goes down.



Connections to points in an annulus

- Points are distributed in the plane by a Poisson process with density λ. Each node is connected to all the nodes inside an annulus $A(r)$ with inner radius r and area 1.
- Theorem: for any critical density λ, one can find a r such that any density above the threshold percolates.



Summary

- Percolation: examines the relationship between local connection v.s. global properties.
- How to check the connectivity?
- Next class: local connectivity rules that guarantee global properties.

Final project

- Project presentation: May 5th.
  - Cover your algorithm and performance analysis (simulation or analytical analysis)

- The final project report is due May 8th.