Matriks dan Ruang Vektor

Operasi Aljabar Matriks

Matriks Invers

Definisi :

Bila A.B = B.A = I, maka A dan B saling invers

Notasi invers A adalah A^{-1}

Sifat-sifat Matriks Invers

Jika A dan B non singular, atau invertibel, maka:

A.B juga non singular

$$(AB)^{-1} = B^{-1}.A^{-1}$$

A matriks bujur sangkar, maka :

$$A^{n} = \{A.A.A. \,..\, A\} \rightarrow \text{n faktor}$$

$$A^{0} = I$$

$$A^{-n} = \left(A^{-1}\right)^{n} = \{A^{-1}.A^{-1}.A^{-1} \,..\, A^{-1}\} \rightarrow \text{n faktor}$$

$$\left(A^{-1}\right)^{-1} = A$$

$$(p.A)^{-1} = p^{-1}.A^{-1} = 1/p\,A^{-1}$$

$$A^n.A^m = A^{n+m}$$

$$\left(A^n\right)^m = A^{n.m}$$

Contoh : A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow A^{-1} = ?$

$$AA^{-1} = I$$

Misalkan $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a+2c = 1   b+2d = 0
3a+4c= 0   3b+4d= 1

a+2c =1 | x2 | →2a+4c =2
3a+ 4c=0 | x1 | →3a+4c =0 -

-a =2

a = -2

3a + 4c = 0

4c = -3

$$c = \frac{-3a}{4} = \frac{-3(-2)}{4}$$

$$c = \frac{3}{2} = 1\frac{1}{2}$$

b+2d =0	x2	→ 2b+4d =0
3b+4d =1	x1	→ 3b+4d =1 -

-b = -1

<u>b = 1</u>

b + 2d = 0.

2d = -b

$$d = \frac{-b}{2} = \frac{-1}{2} = -\frac{1}{2}$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1\,1/2 & -1/2 \end{bmatrix}$$

atau

$$A^{-1} = \frac{1}{/\ A\ /}\,\text{adj (A)}$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 1\ 1/2 & -1/2 \end{bmatrix}$$

Di mana |A|= 1x4-2x3 = -2

1. Rumus penyelesaian Matriks Invers

$$A.\,A^{-1} = \mathrm{I}$$

2. $$(A\,/\,\mathrm{I}) \xrightarrow{\text{OBE}} (\mathrm{I}\,/\,\mathrm{A}^{-1})$$

3. $$A^{-1} = \frac{1}{/\,\mathrm{A}\,/}\,.\ \mathrm{adj}\ (\mathrm{A})$$

Matriks Transpose

Matriks transpose diperoleh dengan menukar elemen-elemen baris men-jadi elemen-elemen kolom dan se-baliknya.

Contoh :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Transpose dari A adalah :

$$A^t - \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Sifat-sifat matriks transpose

1. $\left(A^t\right)^t = A$
2. $(A + B)^t = A^t + B^t$
3. $(p \cdot A)^t = p \cdot A^t$
4. $(A \cdot B)^t = B^t \cdot A^t$

Contoh pembuktian sifat matriks transpose :

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \; dan \;\; B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

Maka $A^t = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \; dan \;\; B^t = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Pembuktian sifat 1:

$$\left[A^t\right]^t = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = A$$

Pembuktian sifat 2 :

$$A + B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 6 \end{bmatrix}, maka \; (A+B)^t = \begin{bmatrix} 5 & 4 \\ 5 & 6 \end{bmatrix}^t = \begin{bmatrix} 5 & 5 \\ 4 & 6 \end{bmatrix}$$

$$A^t + B^t = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 6 \end{bmatrix}$$

Terbukti bahwa $(A+B)^t = A^t + B^t$

Contoh pembuktian sifat 3 :

$$5A = 5\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 5 & 20 \end{bmatrix}, maka\ (5A)^t = \begin{bmatrix} 10 & 15 \\ 5 & 20 \end{bmatrix}^t = \begin{bmatrix} 10 & 5 \\ 15 & 20 \end{bmatrix}$$

$$5A^t = 5\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 15 & 20 \end{bmatrix}$$

Terbukti bahwa

$$(5A)^t - 5A^t$$

Contoh pembuktian sifat 4 :

$$A.\mathrm{B} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6+12 & 2+6 \\ 3+16 & 1+8 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 19 & 9 \end{bmatrix}$$

$$maka\,(\mathrm{A.B})^{\mathrm{t}} = \begin{bmatrix} 18 & 19 \\ 8 & 9 \end{bmatrix}$$

$$B^t.A^t = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6+12 & 3+16 \\ 2+6 & 1+8 \end{bmatrix} = \begin{bmatrix} 18 & 19 \\ 8 & 9 \end{bmatrix}$$

Terbukti bahwa $(A.B)^t = B^t.A^t$

Sifat matriks bujur sangkar A

$A + A^t$ adalah symetric

$A - A^t$ adalah skew symetric

3. A dapat ditulis sebagai jumlah dari suatu matriks symetric B = 1/2 $(A + A^t)$ dan suatu matriks skew symetric C = 1/2 $(A - A^t)$

Soal Latihan :

Tentukan Transpose Suatu Matriks dibawah ini !

1. $$A=\begin{bmatrix}1 & 2 & 0\\ 2 & 3 & 1\\ 0 & 1 & 1\end{bmatrix}, maka:\ A^t = ...$$

2. $$A=\begin{bmatrix}0 & 1 & -1 & -2\\ -1 & 0 & 3 & -4\\ 1 & -3 & 0 & 1\\ 2 & 4 & -1 & 0\end{bmatrix}, maka:\ A^t = ...$$

3. $$A=\begin{bmatrix}2 & 1 & 2\\ 3 & 0 & 1\end{bmatrix}, maka:\ A^t = ...$$

Matriks Eselon dan Matriks Eselon tereduksi

Definisi : $A = [\text{adj}]_{m \times m}$ disebut matriks tereduksi bila memenuhi :

1. Bila ada baris yang tak semua nol, maka elemen pertama yang $\neq 0$ harus bilangan 1
2. Elemen pertama yang $\neq 0$ pada baris dibawahnya harus disebelah kanan 1
3. Baris yang semua nol harus pada bagian bawah (baris-baris bawah)

Matriks Eselon (Eliminasi Gauss)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matriks Eselon Tereduksi (Eliminasi Gauss Jordan):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Contoh Matriks Eselon

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

Contoh Matriks Eselon Tereduksi

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operasi Baris Elementer (OBE)

Definisi :

b_{ij} = menukar baris ke i dengan baris ke j

$b_i(p)$ = mengalikan baris ke i dengan p

$b_{ij}(p) = b_i + p.b_j$

Ganti baris ke i dengan baris baru yang merupakan baris ke i ditambah dengan baris ke j yang dikalikan dengan p.

Contoh :

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 5 & 7 \end{bmatrix} b_{12} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 0 & 5 & 7 \end{bmatrix} b_{2(3)} \begin{bmatrix} 4 & 5 & 6 \\ 3 & 6 & 9 \\ 0 & 5 & 7 \end{bmatrix}$$

$$b_{23}(4)$$

$$b_2 = b_2 + 4.b_3$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 3 & 26 & 37 \\ 0 & 5 & 7 \end{bmatrix}$$

$$\begin{array}{lcrrr} b_2 & = & 3 & 6 & 9 \\ 4b_3 & = & 0 & 20 & 28 \\ \hline & & 3 & 26 & 37 \end{array} +$$

Matriks Elementer dan sifat-sifatnya :

Definisi :

A_{nxn} disebut matriks elementer, bila dengan sekali melakukan OBE terhadap I_n di peroleh A_{nxn}

Contoh :

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{b_2(5)} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{b_2(1/5)} I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{b_{12}} E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{b_{12}} I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[b_3 = b_3 + 4.b_2]{b_{32}(4)} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[b_3 = b_3 + (-4)b_2]{b_{32}(-4)} I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E = Matriks elementer, maka E.A = matriks baru yang terjadi bila OBE tersebut dilakukan pada matriks A

$$A \xrightarrow{OBE} = E.A$$

$$= [I \xrightarrow{OBE}]A$$

Contoh :

$$A - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{b_{12}} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{b_{12}} E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$E.A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Setiap Matriks Elementer adalah matriks tak singular.

Invers matriks elementer juga matriks elementer.

$I \xrightarrow{OBE} E$

maka E^{-1} juga elementer

Cara penyelesaian invers matriks dengan OBE.

$$(A \mid I) \xrightarrow{OBE} (I \mid A^{-1})$$

Contoh 1:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad , maka: \ A^{-1} = ?$$

Solusi :

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{b_{21}(-3)} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \xrightarrow{b_2(-\frac{1}{2})}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow{b_{12}(-2)} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Jadi

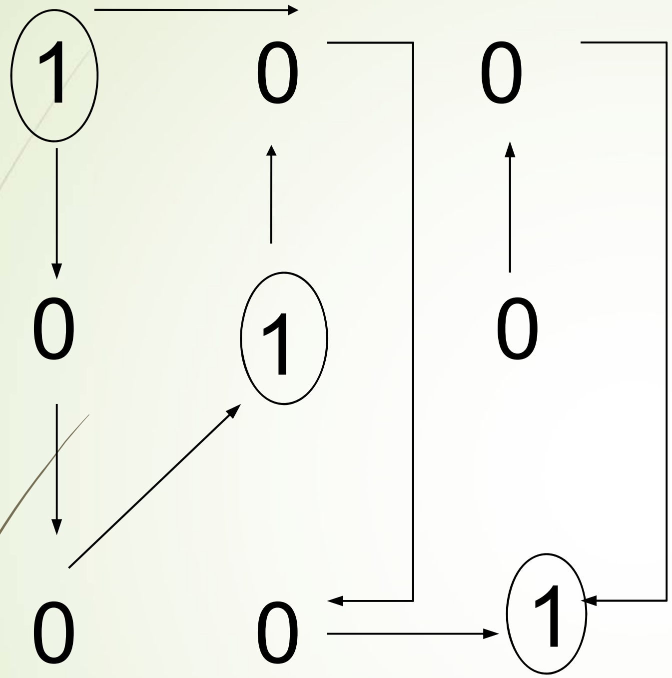
$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 1\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Contoh 2 :

$$B = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 8 & 8 \end{bmatrix}, maka\ B^{-1} = ?$$

Solusi :

$(B \mid I)$ OBE $(I \mid B^{-1})$



$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 8 & 6 & 0 & 1 & 0 \\ 2 & 8 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{b_1(1/2)} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 8 & 6 & 0 & 1 & 0 \\ 2 & 8 & 8 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{b_{21}(-2)} \\ \xrightarrow{b_{31}(-2)} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{b_2(1/2)} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{b_{12}(-3)} \\ \xrightarrow{b_{32}(-2)} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \xrightarrow{b_3(1/2)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{b_{13}(-3)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

I_3 $\quad$ B^{-1}

Jadi $B^{-1} = \begin{bmatrix} 2 & 0 & -1\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Matriks yang tidak mempunyai invers

Contoh :

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[b_{31}(-1)]{b_{21}(-2)} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{b_{23}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & -3 & -3 & -2 & 1 & 0 \end{array}\right] \xrightarrow[b_{32}(3)]{b_{12}(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -5 & 1 & 3 \end{array}\right]$$

Sebelah kiri bukan matriks identitas, maka Matriks B tak mempunyai invers.

Soal latihan :

1) Cari invers matriks dari

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

2) Cari invers matriks dari

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$