SISTEM PERSAMAAN DIFERENSIAL BIASA

Beberapa persamaan diferensial biasa yang berlaku secara simultan disebut Sistem Persamaan Diferensial Biasa (SPDB). Bentuk dari SPDB adalah sebagai berikut :

$$y'_1 = \frac{dy_1}{dx} = f_1(x, y_1, y_2, y_3, ..., y_n) \; ; \; y_1(x_o) = y_{1o}$$

$$y'_2 = \frac{dy_2}{dx} = f_2(x, y_1, y_2, y_3, ..., y_n) \; ; \; y_2(x_o) = y_{2o}$$

$$y'_3 = \frac{dy_3}{dx} = f_3(x, y_1, y_2, y_3, ..., y_n) \; ; \; y_3(x_o) = y_{3o}$$

….

$$y'_n = \frac{dy_n}{dx} = f_n(x, y_1, y_2, y_3, ..., y_n) \; ; \; y_n(x_o) = y_{no}$$

SPDB tersebut dapat ditulis dalam bentuk notasi vektor sebagai berikut :

$$y' = f(x, y); y(x_o) = y_o$$

dimana :

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ ... \\ y_n \end{bmatrix} \; ; \; y' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ ... \\ y_n' \end{bmatrix} \; ; \; f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ ... \\ f_n \end{bmatrix} \; ; \; y_o = \begin{bmatrix} y_{1o} \\ y_{2o} \\ y_{3o} \\ ... \\ y_{no} \end{bmatrix}$$

Solusi dari SPDB tersebut dapat menggunakan metode solusi pada persamaan diferensial tunggal seperti Metode Euler atau Metode Heun yang sudah dibahas sebelumnya.

Contoh 1 :

Diketahui SPDB berikut :

$$\begin{cases} \dfrac{dw}{dx} = -0.5w \; ; \; w(0) = 4 \\ \dfrac{dz}{dx} = 4 - 0.3z - 0.1w; z(0) = 6 \end{cases}$$

Tentukan w dan z menggunakan Metode Euler pada interval $0 \le x \le 1$ dengan Δx=0.5.

Jawab :

$$y = \begin{bmatrix} w \\ z \end{bmatrix} \; ; \; y' = \begin{bmatrix} w' \\ z' \end{bmatrix} \; ; \; f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -0.5w \\ 4 - 0.3z - 0.1w \end{bmatrix} \; ; \; y_o = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Metode Euler : $\begin{aligned} w_{i+1} &= w_i + f_1(x_i, w_i, z_i)\Delta x \\ z_{i+1} &= z_i + f_2(x_i, w_i, z_i)\Delta x \end{aligned}$

$w(0) = 4 \rightarrow x_0 = 0; w_0 = 4$

$z(0) = 6 \rightarrow x_0 = 0; z_0 = 6$

$\Delta x = 0.5$

x_i	0	0.5	1
w_i	4	?	?
z_i	6	?	?

$$\begin{aligned} w_1 &= w_0 + f_1(x_0, w_0, z_0)\Delta x \\ &= 4 + f_1(0,4,6)\Delta x \\ &= 4 + (-0.5 * 4)(0.5) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} w_2 &= w_1 + f_1(x_1, w_1, z_1)\Delta x \\ &= 3 + f_1(0.5;3;6.9)\Delta x \\ &= 3 + (-0.5 * 3)(0.5) \\ &= 3 - 0.75 \\ &= 2.25 \end{aligned}$$

$$\begin{aligned} z_1 &= z_0 + f_2(x_0, w_0, z_0)\Delta x \\ &= 6 + f_2(0,4,6)(0.5) \\ &= 6 + (4 - 0.3 * 6 - 0.1 * 4)(0.5) \\ &= 6 + (4 - 1.8 - 0.4)(0.5) \\ &= 6 + 0.9 \\ &= 6.9 \end{aligned}$$

$$\begin{aligned} z_2 &= z_1 + f_2(x_1, w_1, z_1)\Delta x \\ &= 6.9 + f_2(0.5;3;6.9)(0.5) \\ &= 6.9 + (4 - 0.3 * 6.9 - 0.1 * 3)(0.5) \\ &= 6.9 + (4 - 2.07 - 0.3)(0.5) \\ &= 6.9 + 0.815 \\ &= 7.715 \end{aligned}$$

Hasil :

x_i	0	0.5	1
w_i	4	**3**	**2.25**
z_i	6	**6.9**	**7.715**

Contoh 2 :

Diketahui SPDB berikut :

$$\begin{cases} \dfrac{dx}{dt} = x + z; x(1) = 2 \\ \dfrac{dz}{dt} = xz; z(1) = 4 \end{cases}$$

Tentukan x dan z menggunakan Metode Euler pada interval $1 \le t < 2$ dengan $\Delta t = 0.25$.

Jawab :

$$y = \begin{bmatrix} x \\ z \end{bmatrix} \; ; \; y' = \begin{bmatrix} x' \\ z' \end{bmatrix} \; ; \; f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x + z \\ xz \end{bmatrix} \; ; \; y_o = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Metode Euler : $\begin{aligned} x_{i+1} &= x_i + f_1(t_i, x_i, z_i)\Delta t \\ z_{i+1} &= z_i + f_2(t_i, x_i, z_i)\Delta t \end{aligned}$

$x(1) = 2 \rightarrow t_0 = 1; x_0 = 2$

$z(1) = 4 \rightarrow t_0 = 1; z_0 = 4$

$\Delta t = 0.25$

t_i	1	1.25	1.5	1.75
x_i	2	?	?	?
z_i	4	?	?	?

$$\begin{aligned}
x_1 &= x_0 + f_1(t_0, x_0, z_0)\Delta t \\
&= 2 + f_1(1;2;4)(0.25) \; ; \; f_1(1;2;4) = 2 + 4 = 6 \\
&= 2 + (6)(0.25) \\
&= 2 + 1.5 \\
&= 2.5 \\
z_1 &= z_0 + f_2(t_0, x_0, z_0)\Delta t \\
&= 4 + f_2(1;2;4)(0.25) \; ; \; f_2(1;2;4) = 2 * 4 = 8 \\
&= 4 + (8)(0.25) \\
&= 6 + 2 \\
&= 8
\end{aligned}$$

$$\begin{aligned}
x_2 &= x_1 + f_1(t_1, x_1, z_1)\Delta t \\
&= 2.5 + f_1(1.25;2.5;8)(0.25) \; ; \; f_1(1.25;2.5;8) = 2.5 + 8 = 10.5 \\
&= 2.5 + (10.5)(0.25) \\
&= 2.5 + 2.625 \\
&= 5.125 \\
z_2 &= z_1 + f_2(t_1, x_1, z_1)\Delta t \\
&= 8 + f_2(1.25;2.5;8)(0.25) \; ; \; f_2(1.25;2.5;8) = 2.5 * 8 = 20 \\
&= 8 + (20)(0.25) \\
&= 8 + 5 \\
&= 13
\end{aligned}$$

$$
\begin{aligned}
x_3 &= x_2 + f_1(t_2, x_2, z_2)\Delta t \\
&= 5.125 + f_1(1.5;5.125;13)(0.25) \;;\; f_1(1.5;5.125;13) = 5.125 + 13 = 18.125 \\
&= 5.125 + (18.125)(0.25) \\
&= 5.125 + 4.53125 \\
&= 9.65625 \\
z_3 &= z_2 + f_2(t_2, x_2, z_2)\Delta t \\
&= 13 + f_2(1.5;5.125;13)(0.25) \;;\; f_2(1.5;5.125;13) = 5.125 * 13 = 66.625 \\
&= 13 + (66.626)(0.25) \\
&= 13 + 16.65625 \\
&= 29.65625
\end{aligned}
$$

Hasil :

t_i	1	1.25	1.5	1.75
x_i	2	**2.5**	**5.125**	**9.65625**
z_i	4	**8**	**13**	**29.65625**

Contoh 3 :

Diketahui model SPDB yang diturunkan dari Hukum Kierchoff berikut :

$$
\begin{cases}
\dfrac{di}{dt} = \dfrac{1}{L}\{-Ri - \dfrac{q}{C} + E_0 \sin(\omega t)\};\; i(0) = 0 \\
\dfrac{dq}{dt} = i \;; q(0) = 0
\end{cases}
$$

Jika diketahui L = 1 henry, C = 0.25 coulomb, E_0 = 1 volt, ω = 2 detik dan R = 0 ohm, simulasikan model tersebut menggunakan metode Euler selama 10 detik dengan satuan pengamatan setiap 2 detik.

Jawab :

L = 1 henry,
C = 0.25 coulomb,
E_0 = 1 volt,
ω = 2 detik
R = 0 ohm
Sehingga modelnya menjadi :

$$
\begin{cases}
\dfrac{di}{dt} = \dfrac{1}{1}\{-0i - \dfrac{q}{0.25} + 1 \sin(2t)\};\; i(0) = 0 \\
\dfrac{dq}{dt} = i \;; q(0) = 0
\end{cases}
$$

$$\begin{cases} \dfrac{di}{dt} = -4q + \sin(2t);\ i(0) = 0 \\ \dfrac{dq}{dt} = i\ ; q(0) = 0 \end{cases}$$

$$y = \begin{bmatrix} i \\ q \end{bmatrix}\ ;\ y' = \begin{bmatrix} i' \\ q' \end{bmatrix}\ ;\ f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -4q + \sin(2t) \\ i \end{bmatrix}\ ;\ y_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Metode Euler : $\begin{matrix} i_{i+1} = i_i + f_1(t_i, i_i, q_i)\Delta t \\ q_{i+1} = q_i + f_2(t_i, i_i, q_i)\Delta t \end{matrix}$

$i(0) = 0 \rightarrow t_0 = 0; i_0 = 0$

$q(0) = 0 \rightarrow t_0 = 0; q_0 = 0$

Δt = 2 (detik)

t_i	0	2	4	6	8	10
i_i	0	?	?	?	?	?
q_i	0	?	?	?	?	?

$$\begin{aligned} i_1 &= i_0 + f_1(t_0, i_0, q_0)\Delta t \\ &= 0 + f_1(0;0;0)(2)\ ;\ f_1(0;0;0) = -4(0) + \sin(2(0)) = 0 \\ &= 0 + (0)(2) \\ &= 0 + 0 \\ &= 0 \\ q_1 &= q_0 + f_2(t_0, i_0, q_0)\Delta t \\ &= 0 + f_2(0;0;0)(2)\ ;\ f_2(0;0;0) = i_0 = 0 \\ &= 0 + (0)(2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
i_2 &= i_1 + f_1(t_1, i_1, q_1)\Delta t \\
&= 0 + f_1(2;0;0)(2)\ ;\ f_1(2;0;0) = -4(0) + \sin(2(2)) = -0.7568 \\
&= 0 + (-0.7568)(2) \\
&= 0 + (-1.15136) \\
&= -1.15136 \\
q_2 &= q_1 + f_2(t_1, i_1, q_1)\Delta t \\
&= 0 + f_2(2;0;0)(2)\ ;\ f_2(2;0;0) = i_1 = 0 \\
&= 0 + (0)(2) \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
i_3 &= i_2 + f_1(t_2, i_2, q_2)\Delta t \\
&= -1.15136 + f_1(4;-1.15136;0)(2)\ ;\ f_1(4;-1.15136;0) = -4(0) + \sin(2(4)) = 0.989358 \\
&= -1.15136 + (0.989358)(2) \\
&= -1.15136 + (1.978716) \\
&= 0.465112 \\
q_3 &= q_2 + f_2(t_2, i_2, q_2)\Delta t \\
&= 0 + f_2(4;-1.15136;0)(2)\ ;\ f_2(4;-1.15136;0) = -1.15136 \\
&= 0 + (-1.15136)(2) \\
&= 0 + (-3.02721) \\
&= -3.02721
\end{aligned}
$$

$$
\begin{aligned}
i_4 &= i_3 + f_1(t_3, i_3, q_3)\Delta t \\
&= 0.465112 + f_1(6;0.465112;-3.02721)(2)\ ;\ f_1(6;0.465112;-3.02721) \\
&\qquad = -4(-3.02721) + \sin(2(6)) \\
&\qquad = 12.10884 - 0.5366 = 11.57227 \\
&= 0.465112 + (11.57227)(2) \\
&= 0.465112 + (23.14453) \\
&= 23.60965 \\
q_4 &= q_3 + f_2(t_3, i_3, q_3)\Delta t \\
&= -3.02721 + f_2(6;0.465112;-3.02721)(2)\ ;\ f_2(6;0.465112;-3.02721) = 0.465112 \\
&= -3.02721 + (0.465112)(2) \\
&= -3.02721 + (0.930224) \\
&= -2.09669
\end{aligned}
$$

$$
\begin{aligned}
i_5 &= i_4 + f_1(t_4, i_4, q_4)\Delta t \\
&= 23.60965 + f_1(8;23.60965;-2.09669)(2)\ ; \quad f_1(8;23.60965;-2.09669) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -4(-2.09669) + \sin(2(8)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 8.387948 - 0.2879 = 8.100048 \\
&= 23.60965 + (8.100048)(2) \\
&= 23.60965 + 16.200096 \\
&= 39.809746 \\
q_5 &= q_4 + f_2(t_4, i_4, q_4)\Delta t \\
&= -2.09669 + f_2(8;23.60965;-2.09669)(2)\ ; \ f_2(8;23.60965;-2.09669) = 23.60965 \\
&= -2.09669 + (23.60965)(2) \\
&= -2.09669 + (47.2193) \\
&= 45.12261
\end{aligned}
$$

Hasil

t_i	0	2	4	6	8	10
i_i	0	**0**	**-1.15136**	**0.465112**	**23.60965**	**39.809746**
q_i	0	**0**	**0**	**-3.02721**	**-2.09669**	**45.12261**

Contoh 4 : SPDB dengan 3 PDB

Diketahui SPDB berikut :

$$
\begin{cases}
\dfrac{dx}{dt} = z ; x(0) = 0 \\
\dfrac{dz}{dt} = w ; z(0) = 0.5 \\
\dfrac{dw}{dt} = t - x^2 + z - 3w \ ; \ w(0) = 1
\end{cases}
$$

Tentukan x, z dan w menggunakan Metode Euler pada interval $0 \le t < 0.3$ dengan $\Delta t = 0.1$.

Jawab :

$$
y = \begin{bmatrix} x \\ z \\ w \end{bmatrix} \ ; \ y' = \begin{bmatrix} x' \\ z' \\ w' \end{bmatrix} \ ; \ f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} z \\ w \\ t - x^2 + y - 3w \end{bmatrix} \ ; \ y_o = \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}
$$

$$
\text{Metode Euler :} \quad
\begin{aligned}
x_{i+1} &= x_i + f_1(t_i, x_i, z_i, w_i)\Delta t \\
z_{i+1} &= z_i + f_2(t_i, x_i, z_i, w_i)\Delta t \\
w_{i+1} &= w_i + f_2(t_i, x_i, z_i, w_i)\Delta t
\end{aligned}
$$

$x(0) = 0 \rightarrow t_0 = 0; x_0 = 0$

$z(0) = 0.5 \rightarrow t_0 = 0; z_0 = 0.5$

$w(0) = 1 \rightarrow t_0 = 0; w_0 = 1$

$\Delta t = 0.1$

t_i	0	0.1	0.2	0.3
x_i	0	?	?	?
z_i	0.5	?	?	?
w_i	1			

$$
\begin{aligned}
x_1 &= x_0 + f_1(t_0, x_0, z_0, w_0)\Delta t \\
&= 0 + f_1(0;0;0.5,1)(0.1)\ ;\ f_1(0;0;0.5;1) = 0.5 \\
&= 0 + (0.5)(0.1) \\
&= 0 + 0.05 \\
&= 0.05 \\
z_1 &= z_0 + f_2(t_0, x_0, z_0, w_0)\Delta t \\
&= 0.5 + f_2(0;0;0.5,1)(0.1)\ ;\ f_2(0;0;0.5;1) = 1 \\
&= 0.5 + (1)(0.1) \\
&= 0.5 + 0.1 \\
&= 0.6 \\
w_1 &= w_0 + f_3(t_0, x_0, z_0, w_0)\Delta t \\
&= 1 + f_3(0;0;0.5,1)(0.1)\ ;\ f_3(0;0;0.5;1) = 0 - (0^2) + 0.5 - 3(1) = -2.5 \\
&= 1 + (-2.5)(0.1) \\
&= 1 - 0.25 \\
&= 0.75
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 + f_1(t_1, x_1, z_1, w_1)\Delta t \\
&= 0.05 + f_1(0.1; 0.05; 0.6, 0.75)(0.1) \;;\; f_1(0.1; 0.05; 0.6, 0.75) = 0.6 \\
&= 0.05 + (0.6)(0.1) \\
&= 0.05 + 0.06 \\
&= 0.11 \\
z_2 &= z_1 + f_2(t_1, x_1, z_1, w_1)\Delta t \\
&= 0.6 + f_2(0.1; 0.05; 0.6, 0.75)(0.1) \;;\; f_2(0.1; 0.05; 0.6, 0.75) = 0.75 \\
&= 0.6 + (0.75)(0.1) \\
&= 0.6 + 0.075 \\
&= 0.675 \\
w_2 &= w_1 + f_3(t_1, x_1, z_1, w_1)\Delta t \\
&= 0.75 + f_3(0.1; 0.05; 0.6, 0.75)(0.1) \;; \\
&\qquad f_3(0.1; 0.05; 0.6, 0.75) = 0.1 - (0.05^2) + 0.6 - 3(0.75) = -1.5525 \\
&= 0.75 + (-1.5525)(0.1) \\
&= 0.75 - 0.15525 \\
&= 0.59475
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= x_2 + f_1(t_2, x_2, z_2, w_2)\Delta t \\
&= 0.11 + f_1(0.2; 0.11; 0.675, 0.59475)(0.1) \;;\; f_1(0.2; 0.11; 0.675, 0.59475) = 0.675 \\
&= 0.05 + (0.675)(0.1) \\
&= 0.05 + 0.0675 \\
&= 0.1775 \\
z_3 &= z_2 + f_2(t_2, x_2, z_2, w_2)\Delta t \\
&= 0.675 + f_2(0.2; 0.11; 0.675, 0.59475)(0.1) \;;\; f_2(0.2; 0.11; 0.675, 0.59475) = 0.59475 \\
&= 0.675 + (0.59475)(0.1) \\
&= 0.675 + 0.059475 \\
&= 0.734475 \\
w_3 &= w_2 + f_3(t_2, x_2, z_2, w_2)\Delta t \\
&= 0.59475 + f_3(0.2; 0.11; 0.675, 0.59475)(0.1) \;; \\
&\qquad f_3(0.2; 0.11; 0.675, 0.59475) = 0.2 - (0.11^2) + 0.675 - 3(0.59475) = -0.92135 \\
&= 0.59475 + (-0.92135)(0.1) \\
&= 0.59475 - 0.092135 \\
&= 0.502615
\end{aligned}
$$

PERSAMAAN DIFERENSIAL BIASA ORDE LANJUT

Persamaan diferensial biasa orde lanjut adalah persamaan diferensial biasa dengan orde lebih dari 1. Persamaan diferensial ini dapat ditulis kembali sebagai sistem persamaan diferensial biasa orde 1 dengan membentuk sistem persamaan diferensial biasa.

Misalnya diberikan PDB Orde 2 :

$$y'' = f(x, y, y') \; ; \; y(x_0) = y_0 \;\; dan \; y'(x_0) = z_0$$

Untuk mengubah PDB Orde 2 menjadi Orde 1, dilakukan sebagai berikut :
Ambil $y' = z$ maka $z' = y'' = f(x, y, y'), y(x_0) = y_0 \; dan \, z(x_0) = z_0$. Sehingga PDB Orde 2 di atas dapat ditulis kembali sebagai SPDB berikut :

$$y' = \frac{dy}{dx} = z \; ; \; y(x_0) = y_0$$

$$z' = \frac{dz}{dx} = f(x, y, y') \; ; \; z(x_0) = z_0$$

Selanjutnya, untuk mencari solusi PDB Orde 2 dapat menggunakan metode yang digunakan pada pencarian solusi SPDB yang sudah dibahas sebelumnya.

Contoh :

1. Ubahlah PDB Orde 2 berikut menjadi SPDB :
   a. $y'' - 3y' - 2y = 0 \; ; \; y(0) = 1 \; ; \; y'(0) = 0.5$
   b. $\dfrac{d^2y}{dx^2} - x^2 y \dfrac{dy}{dx} + xy^2 = 0 \; ; \; y(0) = 0.5 \; ; \; y'(0) = 1$
   c. $2\dfrac{d^2s}{dt^2} - 5\dfrac{ds}{dt} - 3s = 45e^{2t} \; ; \; s(0.5) = 2 \; ; \; s'(0.5) = 1$
   d. $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} - E_0 \sin(\omega t) = 0 \; ; \; q(0) = 0 \; ; \; q'(0) = 0$
   e. $\dfrac{d^2x}{dt^2} = \dfrac{1}{M}(F - N - b\dfrac{dx}{dt}) \; ; \; x(0) = 0 \; ; \; x'(0) = 0$
2. Selesaikan PDB Orde 2 pada no. 1a, dan 1b. pada interval $0 \le x \le 1$ dengan Δx =0.25.
3. Diketahui PDB Orde 2 berikut :

   $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} - E_0 \sin(\omega t) = 0 \; ; \; q(0) = 0 \; ; \; q'(0) = 0$$

   Jika diketahui L = 2 henry, C = 0.5 coulomb, E_0 = 2 volt, ϖ = 1.8708 detik, dan R=1 ohm, lakukan simulasi selama 5 detik dengan periode pengamatan setiap 0.5 detik.