Pertemuan 20 s/d 21
INVERS TRANSFORMASI LAPLACE DAN SIFAT-SIFATNYA

Jika Transformasi Laplace dari F(t) adalah f(s) : L[F(t)]=f(s) maka F(t) disebut invers Transformasi Laplace atau kebalikan dari Transformasi Laplace yang dinotasikan dengan

$$F(t) = L^{-1}[f(s)]$$

Contoh :

$L[e^{at}] = \frac{1}{s-a}$, maka $L^{-1}[\frac{1}{s-a}] = e^{at}$

Rumus Invers Transformasi Laplace

1. $L^{-1}\left[\frac{1}{s}\right] = 1$; s > 0

   Bukti :

   $L[1] = L[t^0] = \frac{0!}{s^{0+1}} = \frac{1}{s}$, maka $L^{-1}[\frac{1}{s}] = 1$

2. $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}$; $s > 0; n > -1; \Gamma(n+1) = n\Gamma n = n!$

   Bukti :

   $L[t^n] = \frac{n!}{s^{n+1}}$, maka $L^{-1}[\frac{n!}{s^{n+1}}] = t^n$

   $$n!L^{-1}[\frac{1}{s^{n+1}}] = t^n$$

   $$L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$$

   $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$, maka $L^{-1}\left[\frac{\Gamma(n+1)}{s^{n+1}}\right] = t^n$

$$\Gamma(n+1).L^{-1}\left[\frac{1}{s^{n+1}}\right] = t^n$$

$$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{\Gamma(n+1)}$$

Jadi $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}$; s > 0; n > -1; $\Gamma(n+1) = n\Gamma n = n!$

3. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$; s > a

Bukti :

$L[e^{at}] = \frac{1}{s-a}$, maka $L^{-1}[\frac{1}{s-a}] = e^{at}$; s > a

4. $L^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin \omega t$; s > 0

Bukti :

$L[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$

maka $L^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin \omega t$; s > 0

5. $L^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos \omega t$; s > 0

Bukti :

$L[co\omega t] = \frac{s}{s^2+\omega^2}$

maka $L^{-1}\left[\frac{s}{s^2+\omega^2}\right] = co\omega t$; s > 0

5. $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh \text{at}$; s > |a|

Bukti :

$$L[\sinh at] = \frac{a}{s^2 - a^2} \text{ maka } L^{-1}\left[\frac{a}{s^2 - a^2}\right] = \sinh at \qquad ; s > |a|$$

6. $L^{-1}\left[\frac{s}{s^2 - a^2}\right] = \cosh at \qquad ; s > |a|$

Bukti :

$$L[\cosh at] = \frac{s}{s^2 - a^2} \text{ maka } L^{-1}\left[\frac{s}{s^2 - a^2}\right] = \cosh at \qquad ; s > |a|$$

SIFAT INVERS TRANSFORMASI LAPLACE

1. 1. Sifat Kelinearan

Jika $L^{-1}[f_1(s)] = F_1(t)\, dan\ L^{-1}[f_2(s)] = F_2(t)$

Maka $L^{-1}[C_1\, f_1(s) + C_2\, f_2(s)] = C_1 L^{-1}[f_1(s)] + C_2 L^{-1}[f_2(s)] = C_1 F_1(t) + C_2 F_2(t)$

Contoh :

$$L^{-1}[\frac{8}{S^3} - \frac{3S}{S^2+4} + \frac{5}{S+1}] = L^{-1}[\frac{8}{S^3}] - L^{-1}[\frac{3S}{S^2+4}] + L^{-1}[\frac{5}{S+1}]$$

$$= 8L^{-1}[\frac{1}{S^3}] - 3L^{-1}[\frac{S}{S^2+4}] + 5L^{-1}[\frac{1}{S+1}]$$

$$= 8L^{-1}[\frac{1}{S^{2+1}}] - 3L^{-1}[\frac{S}{S^2+4}] + 5L^{-1}[\frac{1}{S-(-1)}]$$

$$= 8.\frac{t^2}{2!} - 3\cos 2t + 5e^{-t}$$

$$= 4t^2 - 3\cos 2t + 5e^{-t}$$

2. Sifat Translasi

Jika $L^{-1}[f(s)] = F(t)$ maka :

a. $L^{-1}[f(s-a)] = e^{at}\ F(t) = e^{at} L^{-1}[f(s)]$

Bukti:

$L[e^{at}\ F(t)] = f(s-a)$

,maka $L^{-1}[f(s-a)] = e^{at}\ F(t) = e^{at} L^{-1}[f(s)]$

Contoh:

$$L^{-1}[\frac{2}{s^2+2s+5}] = L^{-1}[\frac{2}{(s^2+2s+1)+4}]$$

$$= L^{-1}[\frac{2}{(s+1)^2+2^2}]$$

$$= e^{t\cdot} L^{-1}[\frac{2}{s^2+2^2}]$$

$$= e^{t\cdot} \sin 2t$$

b. $L^{-1}[e^{-as} f(s)] = \begin{cases} F(t-a) & ; t>a \\ 0 & ; t<a \end{cases}$

Bukti:

$L[F(t)] = e^{-as}.f(s)$,

maka $L^{-1}[e^{-as} f(s)] = \begin{cases} F(t-a) & ; t>a \\ 0 & ; t<a \end{cases}$

atau $L^{-1}[e^{-as} f(s)] = \mu(t-a).F(t-a)$

Contoh :

$$L^{-1}\left[\frac{6e^{-2s}}{s^4}\right] = L^{-1}[e^{-2s}.\frac{6}{s^4}] =$$

$f(s) = \frac{6}{s^4}$, maka $F(t) = L^{-1}[f(s)] = L^{-1}\left[\frac{6}{s^4}\right] = 6L^{-1}[\frac{1}{s^4}]$

$$= 6L^{-1}[\frac{1}{s^{3+1}}] = 6.\frac{t^3}{3!} = \frac{6t^3}{6} = t^3$$

$F(t) = t^3$ dan a=2, maka $F(t-2) = (t-2)^3$

Jadi $L^{-1}[\frac{6e^{-2s}}{s^4}] = \mu(t-2).F(t-2) = \mu(t-2).(t-2)^3$

3. Sifat Perubahan Skala

Jika $L^{-1}[f(s)] = F(t)$ maka $L^{-1}\left[f(\frac{s}{a})\right] = a.F(at)$ [XXX $L^{-1}[f(as)] = \frac{1}{a}F\left(\frac{t}{a}\right)$]

Bukti :

$L[F(at)] = \frac{1}{a} f(\frac{s}{a})$, maka $L^{-1}\left[\frac{1}{a} f(\frac{s}{a})\right] = F(at)$

$$\frac{1}{a}.L^{-1}\left[f(\frac{s}{a})\right] = F(at)$$

$$L^{-1}\left[f(\frac{s}{a})\right] = a.F(at)$$

Contoh :

$$L^{-1}\left[\frac{3}{s^2+9}\right] = L^{-1}\left[\frac{\frac{3}{9}}{\frac{s^2+9}{9}}\right] = L^{-1}\left[\frac{\frac{1}{3}}{\frac{s^2}{9}+\frac{9}{9}}\right] =$$

$$= L^{-1}\left[\frac{\frac{1}{3}}{(\frac{s}{3})^2+1}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(\frac{s}{3})^2+1}\right] = \frac{1}{3}.3\sin 3t$$

$$= \sin 3t$$

4. Invers Transformasi Laplace dari Derivatif

*Perkalian dengan t^n

Jika $L^{-1}[f(s)] = F(t)$ maka $L^{-1}\left[f^{(n)}(s)\right] = (-1)^n.t^n.F(t)$

Bukti:

$L[t^n.F(t)] = (-1)^n.f^{(n)}(s)$,

maka $L^{-1}\left[(-1)^n.f^{(n)}(s)\right] = t^n.F(t)$

$$(-1)^n.L^{-1}\left[f^{(n)}(s)\right] = t^n.F(t)$$

$$L^{-1}\left[f^{(n)}(s)\right] = \frac{t^n}{(-1)^n}.F(t)$$

$$L^{-1}\left[f^{(n)}(s)\right] = (-1)^n.t^n.F(t) = (-1)^n.t^n.L^{-1}[f(s)]$$

Contoh :

$$L^{-1}\left[\frac{1}{(s-2)^2}\right]=$$

Jawab:

$f(s)=\dfrac{1}{s-2}$, maka $L^{-1}\left[\dfrac{1}{s-2}\right]=e^{2t}$

$$f(s)=\frac{1}{s-2}=(s-2)^{-1}$$

$$f^{(1)}(s)=(-1).(s-2)^{-1-1}.(s-2)'$$

$$f^{(1)}(s)=(-1).(s-2)^{-2}.(1)=-\frac{1}{(s-2)^2}$$

Menurut Rumus:

$$L^{-1}\left[f^{(n)}(s)\right]=(-1)^n.t^n.F(t)=(-1)^n.t^n.L^{-1}[f(s)]$$

$$L^{-1}\left[f^{(1)}(s)\right]=(-1)^1.t^1.F(t)=(-1)^1.t^1.L^{-1}[f(s)]$$

$$(-1)L^{-1}\left[\frac{1}{(s-2)^2}\right]=(-1)\cdot t\ .L^{-1}[\frac{1}{s-2}]$$

$$L^{-1}\left[\frac{1}{(s-2)^2}\right]=t.e^{2t}$$

5. Invers Transformasi Laplace dari Integral

* **Jika** $L^{-1}[f(s)]=F(t)$ maka $L^{-1}\left[\dfrac{f(s)}{s}\right]=\int_0^t F(u)du$

Bukti:

$L\left[\int_0^t F(u)du\right]=\dfrac{f(s)}{s}$, maka $L^{-1}\left[\dfrac{f(s)}{s}\right]=\int_0^t F(u)du$

* **Perkalian dengan s^n**

Jika $L^{-1}[f(s)]=F(t)$ maka

a. $L^{-1}[s.f(s)]=F'(t)$; $F(0)=0$

Bukti :

$L[\frac{dF}{dt}] = s.L[F(t)] - F(0)$, di mana F(0)=0

$L[\frac{dF}{dt}] = s.f(s) - 0 = s.f(s)$,

maka $L^{-1}[s.f(s)] = \frac{dF}{dt} = F'(t)$

b. $L^{-1}\left[s^2.f(s) - s.F(0)\right] = F''(t) \quad ; F(0) \neq 0$

Bukti :

$L[\frac{d^2F}{dt^2}] = s^2.L[F(t)] - s.F(0)$, di mana $F(0) \neq 0$

$L[\frac{d^2F}{dt^2}] = s^2.f(s) - s.F(0)$,

maka $L^{-1}\left[s^2.f(s) - s.F(0)\right] = F''(t)$

*** Pembagian dengan s**

Jika $L^{-1}\left[f(s)\right] = F(t)$ maka $L^{-1}\left[\frac{f(s)}{s}\right] = \int_o^t F(u)\,du$

Bukti:

$L\left[\int_0^t F(u)du\right] = \frac{f(s)}{s}$, maka $L^{-1}\left[\frac{f(s)}{s}\right] = \int_0^t F(u)du$

*** Sifat Convolution**

Jika $L^{-1}\left[f(s)\right] = F(t) \quad ; L^{-1}\left[g(s)\right] = G(t)$ maka

$$L^{-1}\left[f(s).g(s)\right] = \int_o^t F(u)\,G(t-u)\,du \quad \text{atau}$$

$$= \int_o^t G(u)\,F(t-u)\,du$$

Contoh:

1. $L^{-1}\left[\dfrac{4-5s}{s^{3/2}}\right]=$

Jawab:

$$L^{-1}\left[\frac{4-5s}{s^{3/2}}\right]=L^{-1}\left[\frac{4}{s^{3/2}}-\frac{5s}{s^{3/2}}\right]=4.L^{-1}\left[\frac{1}{s^{3/2}}\right]-5.L^{-1}\left[\frac{1}{s^{1/2}}\right]$$

$$=4.L^{-1}\left[\frac{1}{s^{1/2+1}}\right]-5.L^{-1}\left[\frac{1}{s^{-1/2+1}}\right]$$

$$=4.\frac{t^{1/2}}{\gamma(1/2+1)}-5.\frac{t^{-1/2}}{\gamma(-1/2+1)}$$

$$=\frac{4.t^{1/2}}{1/2.\gamma(1/2)}-\frac{5.t^{-1/2}}{\gamma(1/2)}=\frac{8.t^{1/2}}{\gamma(1/2)}-\frac{5.t^{-1/2}}{\gamma(1/2)}=$$

$$=\frac{8.t^{1/2}}{\gamma(1/2)}-\frac{5.t^{-1/2}}{\gamma(1/2)}=\frac{8.t^{1/2}-5.t^{-1/2}}{\gamma(1/2)}$$

$$=\frac{8.t^{1/2}-5.t^{-1/2}}{\gamma(1/2)}=\frac{8.t^{1/2}-5.t^{-1/2}}{\sqrt{\pi}}$$

2. $L^{-1}[\dfrac{1}{s^2+2s}]=$

Jawab:

$$L^{-1}[\frac{1}{s^2+2s}]=L^{-1}[\frac{1}{s(s+2)}]=$$

$$\frac{1}{s(s+2)}=\frac{A}{s}+\frac{B}{s+2}$$

$$A=s.\frac{1}{s(s+2)}=\frac{1}{0+2}=1/2$$

(di mana s=0)

$$B=(s+2).\frac{1}{s(s+2)}=\frac{1}{-2}=-1/2$$

(di mana s+2 = 0, maka s=-2)

$$L^{-1}[\frac{1}{s^2+2s}] = L^{-1}[\frac{1}{s(s+2)}] = L^{-1}[\frac{A}{s} + \frac{B}{s+2}] =$$

$$= L^{-1}[\frac{A}{s}] + L^{-1}[\frac{B}{s+2}] = L^{-1}[\frac{1/2}{s}] + L^{-1}[\frac{-1/2}{s+2}]$$

$$= 1/2\, L^{-1}[\frac{1}{s}] - 1/2\, L^{-1}[\frac{1}{s-(-2)}]$$

$$= 1/2\,(1) - 1/2\, e^{-2t} = 1/2\,(1-e^{-2t})$$

3. $L^{-1}[\frac{2s+3}{s^2-2s+5}] =$

Jawab:

$s^2-2s+5 = (s^2-2s+1)+4 = (s-1)^2+2^2$

2(s-1)=2s-2

$2s+3 = (2s-2)+5 = 2(s-1)+5$

$$L^{-1}[\frac{2s+3}{s^2-2s+5}] = L^{-1}[\frac{2(s-1)+5}{(s-1)^2+2^2}] = e^{t.} L^{-1}[\frac{2s+5}{s^2+2^2}]$$

$$= e^{t.} L^{-1}[\frac{2s+5}{s^2+2^2}] = e^{t.} \{L^{-1}[\frac{2s}{s^2+2^2}] + L^{-1}[\frac{5}{s^2+2^2}]\}$$

$$= e^{t.} \{2L^{-1}[\frac{s}{s^2+2^2}] + \frac{5}{2} L^{-1}[\frac{2}{s^2+2^2}]\}$$

$$= e^{t.} \{2.\cos 2t + \frac{5}{2}.\sin 2t\}$$

$$= 1/2\, e^{t.} \{4.\cos 2t + 5.\sin 2t\}$$

4. $L^{-1}[\frac{1}{\sqrt{s-3}}] =$

Jawab:

$$L^{-1}[\frac{1}{\sqrt{s-3}}] = L^{-1}[\frac{1}{(s-3)^{1/2}}] = e^{3t}.L^{-1}[\frac{1}{s^{1/2}}] =$$

$$= e^{3t}.L^{-1}[\frac{1}{s^{-\frac{1}{2}+1}}] = e^{3t}.\frac{t^{-\frac{1}{2}}}{\gamma(-\frac{1}{2}+1)} = e^{3t}.\frac{t^{-\frac{1}{2}}}{\gamma(\frac{1}{2})} = \frac{e^{3t}.t^{-\frac{1}{2}}}{\sqrt{\pi}}$$

* PENGGUNAAN TRANFORMASI LAPLACE

Penggunaan Transformasi Laplace umumnya digunakan untuk menyelesaikan persamaan tanpa melakukan manipulasi matematis komplek tetapi menggunakan manipulasi Aljabar Biasa.

Transformasi Laplace dapat digunakan antara lain untuk menyelesaikan permasalahan berikut :

- Bentuk Gelombang Periodik maupun tidak periodik
- Transienitas dalam rangkaian linear
- Sistem linear dengan atau tanpa umpan balik
- Vibrasi transien di dalam sistem mekanik
- Propagasi sinyal dalam sistem komunikasi
- Penyelesaian persamaan diferensial dan lainnya