Penggunaan Integral Lipat 2 dan Integral Lipat 3

Kalkulus 2

Integral lipat dua dapat digunakan dalam banyak hal : misal,

I. Menghitung volume antara permukaan $z = f(x,y)$ dan bidang xy

Rumus :

$$V = \int_R \int f(x,y)\, dx\, dy$$

II. Menghitung luas daerah dibidang xy dimana $f(x,y) = 1$

Rumus :

$$L = \int_R \int dx\, dy$$

III. Menghitung massa

f dipandang sebagai massa jenis (massa persatuan luas)

Rumus :

$$M = \int_R \int f(x,y)\,dx\,dy$$

IV. Menghitung pusat massa

f = massa jenis, M = massa dari pelat tipis dan (x, y) = pusat massa di R maka

$$M_x = \int_R \int x\, f(x,y)\,dx\,dy$$

$$M_y = \int_R \int y\, f(x,y)\,dx\,dy$$

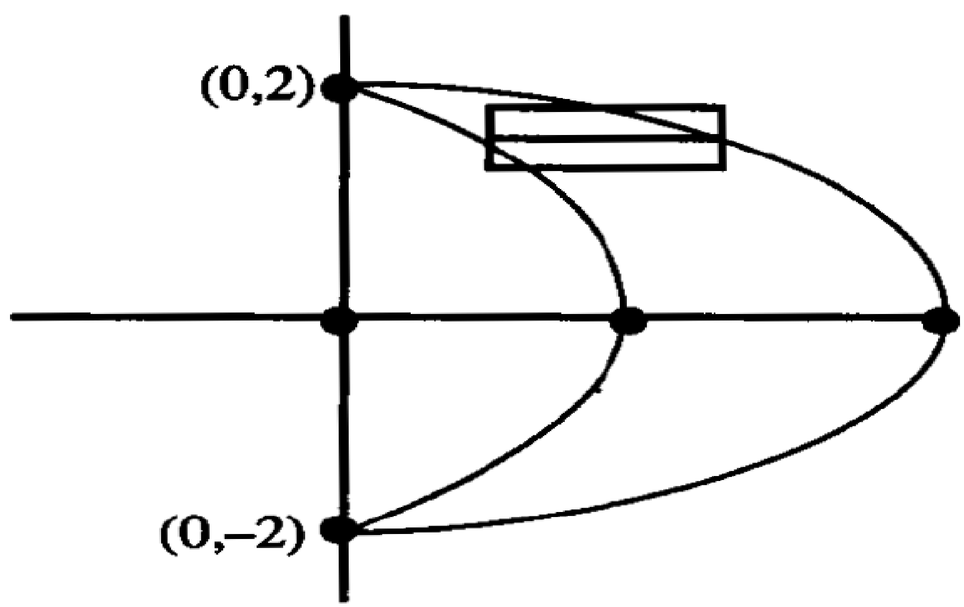
Hitung luas daerah yang dibatasi oleh parabola-parabola

$$y^2 = 4 - x$$
$$y^2 = 4 - 4x$$

Cari titik potong kedua parabola

$$4 - x = 4 - 4x$$
$$3x = 0$$
$$x = 0$$
$$y = \pm 2$$

Titik-titik potong : (0, 2) dan (0, –2)



$$L = 2\int_0^2 \int_{1-\frac{y^2}{4}}^{4-y^2} dx \quad dy$$

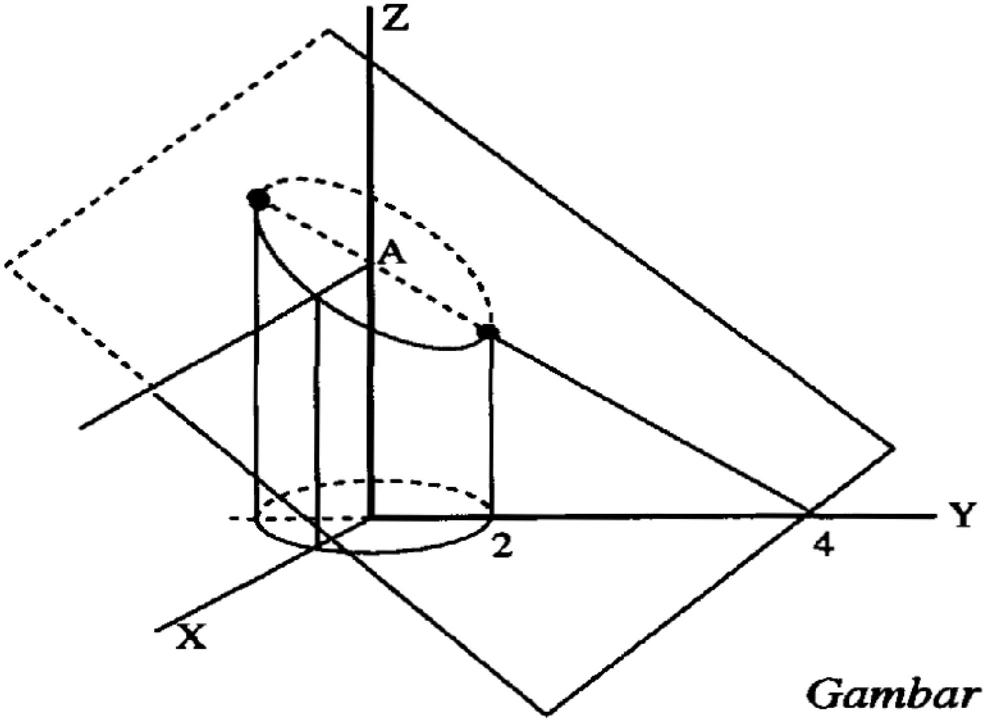
$$= \int_0^2 x \int_{1-\frac{y^2}{4}}^{4-y^2} \quad dy$$

$$= 2\int_0^2 (4 - y^2 - 1 + \frac{y^2}{4}) \, dy$$

$$= 2\ (3y - \frac{1}{4} y^3 \ \Big|_0^2$$

$$= 2\ (6 - 2) = 8$$

Hitung volume ruang yang dibatasi oleh silinder $x^2 + y^2 = 4$ dan bidang-bidang $y + z = 4$ dan $z = 0$



Gambar 6.9

$$V = \int_R \int z \, dA$$

$$V = \int_R \int (4 - y) \, dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4 - y) \, dx \, dy$$

$$= 2 \int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} (4 - y) \, dx \, dy$$

$$V = 2 \int_{-2}^{2} (4x - y\ x) \Big|_{0}^{\sqrt{4-y^2}} dy$$

$$= 2 \left\{ \int_{-2}^{2} (4 (\sqrt{4 - y^2}) - y (\sqrt{4 - y^2}) \, dy \right.$$

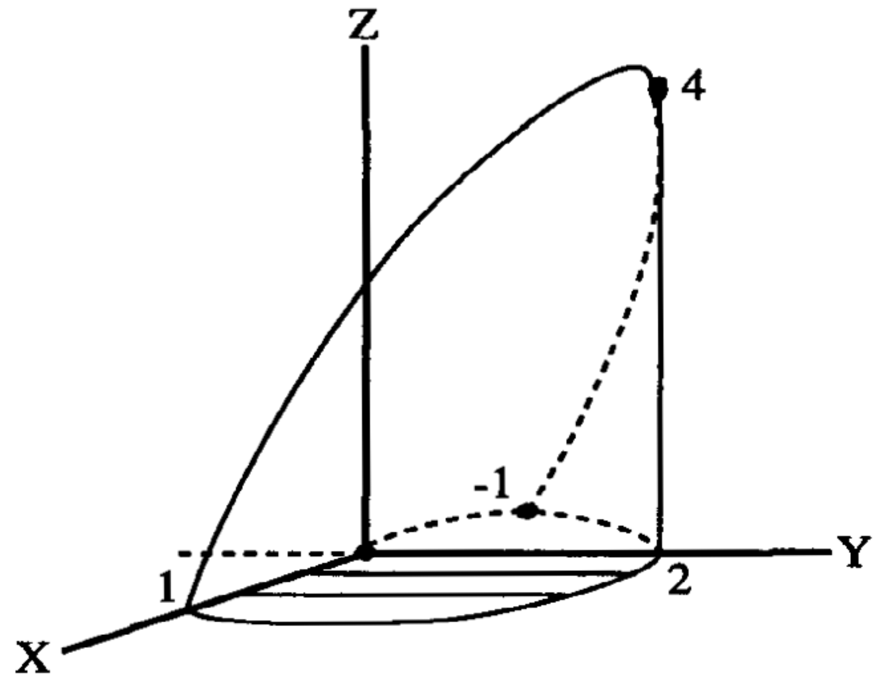
$$= 2.4 \left(\frac{1}{2} y \sqrt{4 - y^2} + \frac{1}{2} . 4 \text{ arc sin } \frac{y}{2} \right) \Big|_{-2}^{2}$$

$$+ {}^{1}/_{2} \int_{-2}^{2} (4 - y^2)^{1/2} \, d (4 - y^2)$$

$$= 4 (0 - 0) + 4 . 4 \left(\frac{\pi}{2} + \frac{\pi}{2}\right) + 2 . \frac{2}{3} (4-y^2)^{3/2} \Big|_{-2}^{2}$$

$$= 16\pi + \frac{4}{3} (0-0) = 16\pi$$

Hitung volume dari ruang yang dibatasi oleh silinder $4x^2 + y^2 = 4$, bidang $z = 0$ dan $z = 2y$



Gambar 6.10

$$V = \int_R \int z \, dA$$

$$= \int_{y=0}^{2} \int_{\frac{-\sqrt{4-y^2}}{2}}^{\frac{\sqrt{4-y^2}}{2}} 2y \, dx \, dy$$

$$= 2\int_0^2 \int_0^{\frac{\sqrt{4-y^2}}{2}} 2y \, dx \, dy$$

$$= 2\int_0^2 2yx \Big|_0^{\frac{\sqrt{4-y^2}}{2}} dy$$

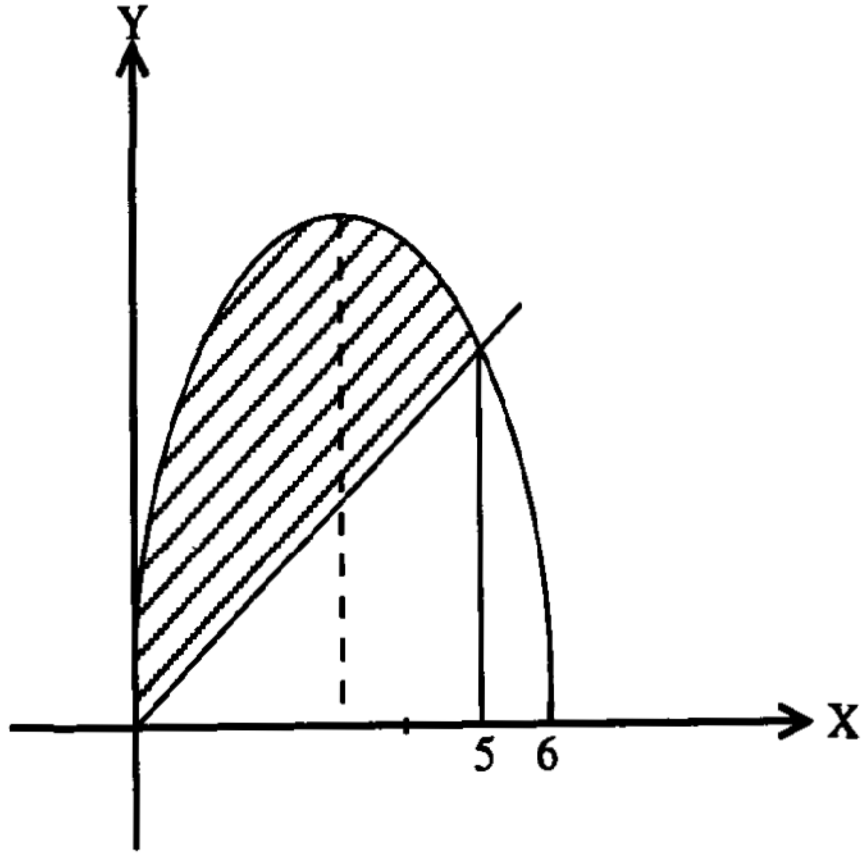
$$= 2\int_0^2 2y \cdot \frac{1}{2}\sqrt{4-y^2}\, dy$$

$$= -\int_0^2 (\sqrt{4-y^2})\, d\,(4-y^2)$$

$$= -\left[\frac{2}{3}(4-y^2)^{3/2}\right]_0^2$$

$$= -\frac{2}{3}(0-8) = \frac{16}{3}$$

Tentukan pusat bidang yang luasnya dibatasi oleh parabola $y = 6x - x^2$ dan $y = x$



Gambar 6.11

Jawab :

$$A = \int_R \int dA = \int_{x=0}^{5} \int_{x}^{6x-x^2} dy\, dx$$

$$= \int_0^5 (6x - x^2 - x)\, dx$$

$$= \frac{5}{2} x^2 - \frac{1}{3} x^3 \Big|_0^5 = \frac{125}{2} - \frac{125}{3} = \frac{125}{6}$$

$$M_x = \int_R \int y\, dA = \int_0^5 \int_x^{6x-x^2} y\, dy\, dx$$

$$= \frac{1}{2} \int_0^5 y^2 \Big|_x^{6x-x^2} dx$$

$$= \frac{1}{2} \int_0^5 \{(6x - x^2)^2 - x^2\}\, dx$$

$$= \frac{1}{2} \int_0^5 (35x^2 - 12x^3 + x^4)\, dx$$

$$= \frac{1}{2} \left(\frac{35}{3} x^3 - 3 x^4 + \frac{1}{5} x^5\right) \Big|_0^5$$

$$= \frac{1}{2} \left(\frac{4375}{3} - 1875 + 625 \right) = \frac{625}{6}$$

$$y = \frac{M_x}{A} = \frac{625/6}{125/6} = 5$$

$$M_y = \int_R \int x\, dA = \int_0^5 \int_x^{6x-x^2} x\, dy\, dx$$

$$= \int_0^5 (xy) \Big|_x^{6x-x^2} dx = \int_0^5 (6x^2 - x^3 - x^2)\, dx$$

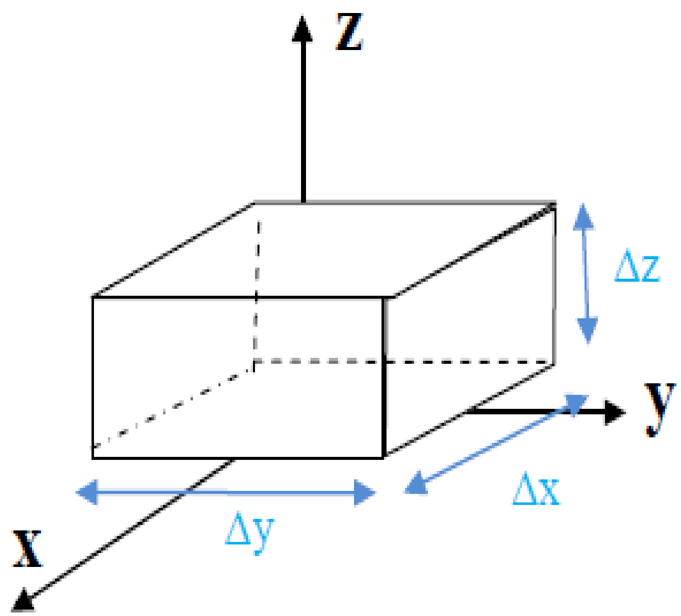
$$= \left[\frac{5}{3} x^3 - \frac{1}{4} x^4 \right]_0^5 = \frac{625}{3} - \frac{625}{4}$$

$$= \frac{625}{12}$$

$$x = \frac{M_y}{A} = \frac{{}^{625}/_{12}}{{}^{125}/_{6}} = \frac{5}{2}$$

Pusat bidang (5/2, 5)

INTEGRAL LIPAT TIGA



Integral lipat tiga

Daerah domain di $\mathbb{R}^3$:

$a_1 \leq x \leq a_2, \phi_1(x) \leq y \leq \phi_2(x), \varphi_1(x,y) \leq z \leq \varphi_2(x,y)$.

Membuat partisi pada daerah domain:

$$\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint f(x,y,z)\,dV$$

$$= \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Hitung $\iiint_S f(x,y,z)\,dV$ dengan W=f(x,y,z) = 2xyz dan S benda padat yang dibatasi oleh tabung parabola $z=2-\frac{1}{2}x^2$ dan bidang-bidang z = 0, y=x, y=0

Jawab.

Dari gambar terlihat bahwa

$S=\{(x,y,z)|0\le x\le 2,\ 0\le y\le x\ \ 0\le z\le 2-\frac{1}{2}x^2\}$

Sehingga,

$$\iiint_S 2xyz\,dV = \int_0^2\int_0^x\int_0^{2-\frac{1}{2}x^2} 2xyz\,dz\,dy\,dx$$

$$= \int_0^2\int_0^x xy\ z^2\Big|_0^{2-\frac{1}{2}x^2} dy\,dx$$

$$= \int_0^2\int_0^x xy\left(2-\frac{1}{2}x^2\right)^2 dy\,dx$$

$$= \int_0^2 x\left(4-2x^2+\frac{1}{4}x^4\right)\ \frac{1}{2}y^2\Big|_0^x dx$$

$$= \int_0^2 \left(2x^3-x^5+\frac{1}{8}x^7\right)\ dx$$

$$= \frac{1}{2}x^4-\frac{1}{6}x^6+\frac{1}{64}x^8\Big|_0^2$$

$$= 8-\frac{32}{3}+4=\frac{4}{3}$$

Jawab.

Dari gambar terlihat bahwa

$S = \{(x,y,z) | 0 \le x \le 2,\ 0 \le y \le x$
$0 \le z \le 2 - \frac{1}{2}x^2\}$

Sehingga,

$$\iiint_S 2xyz\ dV = \int_0^2 \int_0^x \int_0^{2-\frac{1}{2}x^2} 2xyz\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^x xy\ \ z^2 \Big|_0^{2-\frac{1}{2}x^2}\ dy\, dx$$

$$= \int_0^2 \int_0^x xy\left(2 - \frac{1}{2}x^2\right)^2 dy\,dx$$

$$= \int_0^2 x\left(4 - 2x^2 + \frac{1}{4}x^4\right) \frac{1}{2}y^2\Big|_0^x dx$$

$$= \int_0^2 \left(2x^3 - x^5 + \frac{1}{8}x^7\right) dx$$

$$= \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{64}x^8\Big|_0^2$$

$$= 8 - \frac{32}{3} + 4 = \frac{4}{3}$$